Combinatorial Algebraic Topology
Midterm Solutions

1. Give an example of two spaces that are homotopy equivalent but not homeomorphic. (No proof needed.)

Solution

The spaces from question 4!

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2. Describe, with proof, a surjective nullhomotopic map $f : S^n \to S^1$.

Solution

Let $p : S^n \to [-1, 1]$ be the projection onto the first coordinate, and $g : [-1, 1] \to S^1$ be $g(x) = (\cos(x\pi), \sin(x\pi))$. Then $f = g \circ p$ maps S^n surjectively onto S^1. The function $H : I \times S^n : (t, x) \to g(t \cdot p(x))$ is continuous and restricts on $\{0\} \times S^n$ to the constant function $g(0) = (0, 0)$ and on $\{1\} \times S^n$ to f. This shows that f is nullhomotopic.

3. Show that a continuous map $f : S^n \to S^n$ is nullhomotopic if and only if it is homotopic to a continuous map $g : S^n \to S^n$ that is not surjective.

Solution

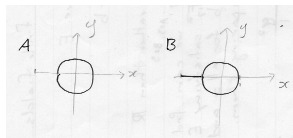
If it is nullhomotopic then it is homotopic to a constant map which is clearly not surjective, so this direction is clear. For the other, direction, it is enough to find a nullhomotopy of g, as the relation of being homotopic is an equivalence. This is simple. The map g misses some point x_0, we may assume x_0 is $(1, 0, 0, \ldots, 0)$. The stereographic projection s of $S^n \setminus \{x_0\}$ onto $\mathbb{R}^n$ is thus a well defined homeomorphism. Letting $H : I \times R^n \to R^n$ be the standard deformation retraction of R^n to the origin, the composition

$$I \times S^n \xrightarrow{\text{id} \times g} I \times S^n \xrightarrow{\text{id} \times s} I \times \mathbb{R}^n \xrightarrow{H} \mathbb{R}^n \xrightarrow{s^{-1}} S^n$$

is a homotopy of g to the constant map to $(-1, 0, 0, \ldots 0)$.

Extra: The stereoprojection s maps x in $S^n \setminus x_0$ to the intersection of the ray from x_0 to x with R^n. The standard deformation retraction H of R^n to the origin is $H : (t, x) \mapsto t \cdot (0, 0, \ldots, 0) + (t - 1)x$.

4. Let $A = S^1$, $B' = \{(x, 0) \mid x \in [-1, -2]\}$ and $B = A \cup B'$.



Find continuous maps $f : A \to B$ and $g : B \to A$ and use them to show that A and B are homotopy equivalent. (This might take some time. If you describe the maps f and g now, you may hand in the proof of homotopy equivalence next Tuesday.)

Solution

Let $z : I \to B$ be a continuous injection of I to B that takes 0 to $(0, 1)$, $1/4$ to $(1, 0)$, $1/2$ to $(0, -1)$ and 1 to $(0, -2)$. (Explicitly, we could take $z : I \to B$ be defined as follows. Let $z(t) = (\cos 2\pi \cdot t, \sin 2\pi \cdot t)$ for $t \leq 1/2$ and $z(t) = (-2t, 0)$ for $t \geq 1/2$.)

Let $f_t : A \to B$ be defined for $t \in [1, 2]$ by

$$f_t((x, y)) = z(t \times z^{-1}((x, y))),$$

if $y \geq 0$ and to be the reflection of $f_t((x, -y))$ in $y = 0$ if $y < 0$. Observe that f_2 is a surjection onto B, and $f_1 = \mathrm{id}_A$. Let $f = f_2$.

Let $g : B \to A$ be the identity on A and take B' to $(-1, 0)$. .

Now $g \circ f_t$ agrees with f_t on A for any t, so $H : [1, 2] \times A \to A : (t, x) \to g \circ f_t$ is a homotopy of $g \circ f_1 = \mathrm{id}_A$ to the map $g \circ f_2 = g \circ f$.

On the other hand $f_1 \circ g = \mathrm{id}_A$ $H : [1, 2] \times B : (t, x) \to f_t \circ g(x)$ is a homotopy of the identity $f_1 \circ g$ to $f_2 \circ g = f \circ g$.

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5. Show how many 1-simplices there are in the barycentric subdivision of an n-simplex σ^n.

Solution

The 0-simplices of $\mathrm{sd}(\sigma^{\mathrm{n}})$ are the non-empty subsets of $[n+1]$, and two such subsets S, T form a 1-simplex if one is contained in the other. For all $i = 1, \ldots, n$, there are $\binom{n+1}{i}$ subsets of $[n+1]$ of cardinality i, and each is contained in $2^{(n+1-i)} - 1)$ supersets in $[n+1]$.
So $\mathrm{sd}(\sigma^{\mathrm{n}})$ has

$$\sum_{i=1}^{n} \binom{n+1}{i} (2^{(n+1-i)} - 1)$$

one simplices. (Bonus points if you get it in a nicer form than I did.)

(If you can't answer the question above, answer the following for partial marks)

5' (a) How many 1-simplices are there in a 3-simplex.

Solution

6

(b) Draw the barycentric subdivision of a 2-simplex.

(c) How many 0-simplices are there in the barycentric subdivision of a 3 simplex.

Solution

$2^4 - 1 = 15$.

(d) How many 1-simplices are there in the barycentric subdivision of a 3-simplex.

Solution

50

enumerate

(e) Prove that the diameter of a simplex σ is the distance between some two vertices of σ.

Solution

You've all seen the solution to this.

(f) Show the equivalence of the following two versions of the Borsuk Ulam Theorem.

[BU2a] There is no continuous antipodal mapping $f : S^n \to S^{n-1}$.

[Bu2b] There is no continuous mapping $g : B^n \to S^{n-1}$ that is antipodal on the boundary.

Solution

In the text.

(g) State Tucker's Lemma (any version) and show that it is implied by the Borsuk Ulam Theorem.

Solution

In the text.