Combinatorial Algebraic Topology - Extra Notes

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This is additional notes for a class taught from Matoušek's 'Using the Borsuk-Ulam Theorem'. The text assumes a working familiarity with topological ideas. This gives some examples to fill in a lack of such familiarity.

1 Chapter 1

Topological spaces are mostly assumed to be compact subspaces of real spaces, with the subspace metric topology. So continuity of a function can be defined with ε and δ.

Definition 1.1. A function $f : X \to Y$ is *continuous* if for all $x \in X$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x' \in X$,

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon.$$

$f : X \to Y$ is *uniformly continuous* if for all $\varepsilon > 0$ there exists s a $\delta > 0$ such that for all $x, x' \in X$,

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \epsilon.$$

In compact spaces continuity and uniform continuity are equivalent.

It is clear that if we define a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ then this function restricted to $X \subset R^n$ is also continuous, we can use the same δ for any ε.

We recall from calculus that any algebraic function in d variables is continuous on its domain. As are the trigonometric functions. So any function f on $X \subset \mathbb{R}$ that we define by a polynomial such as $f(x, y) = x^3y - 5x + y + 8$ is continuous. In calculus you show that the sum, product and composition of continuous functions is continuous. Recall how we did such things:

Proposition 1.2. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is continuous, then so is $h = g \circ f : X \to Z$.*

Proof. It simplifies the proof to replace 'continuous' with 'uniformly continuous' in each occurence. For us this is the same.

Let $\varepsilon > 0$. As g is uniformly continuous there is δ_Y such that for $y, y' \in Y$,

$$|y - y'| < \delta_Y \Rightarrow |g(y) - g(y')| < \varepsilon.$$

As f is uniformly continuous, there is δ such that for $x, x' \in X$,

$$|x - x'| < \delta \Rightarrow |f(x) - f(x')| < \delta_Y.$$

So

$$|x - x'| < \delta \Rightarrow |g \circ f(x) - g \circ f(x')| < \varepsilon.$$

□

This gives us that such functions as $f(x, y) = x \sin^2(x)y - 2x + y/\sin(x)$ are continuous on their domains. (Look out where $\sin(x) = 0$.)

Further, a function $f : R^n \to R^m$ such as $f(x, y) = (x^2, yx)$ is continuous if and only if its component functions $f_1(x, y) = x^2$ and $f_2(x, y) = yx$ are.

Try proving this for and $f(x, y) = (f_1(x, y), f_2(x, y))$. Recall that for $(x, y), (x', y') \in \mathbb{R}^2$,

$$|(x, y) - (x', y')|^2 = |x - x'|^2 + |y - y'|^2.$$

If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are continuous functions, we write $f_1 \times f_2$ for the continuous function

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 : (x_1, x_2) \mapsto (f_1(x_1), f_2(x_2)).$$

One fact that we will use frequently in showing that maps are continuous is the following, which actually holds for all topological spaces. Section 1.1 Exercise 2 leads you through most of the more general proof. For us it is quite easy to do it directly though.

A *cover* of a space X is a family $\{X_i\}_{i \in I}$ of subsets of X whose union is X.

Lemma 1.3 (The Gluing Lemma)**.** *Let $A_1, \ldots, A_n$ be cover of closed sets of X. For each i let $f_i : A_i \to Y$ be a continuous functions such that for all $i, j \in [n]$, f_i and f_j agree on $A_i \cap A_j$. Then the function $\bigcup_i^n f_i = f : X \to Y$ is well defined by*

$$f(x) = f_i(x) \text{ if } x \in A_i,$$

is continuous.

Proof. Let $\varepsilon > 0$, and $x \in X$. Let $I \subset [n]$ be the set of indices i for which $x \in A_i$. For $i \in I$, we have by the continuity of f_i that there is some $\delta_i > 0$ such that

$$\begin{aligned} \forall y \in A_i, |y - x| < \delta_i \quad &\Rightarrow \quad |f_i(y) - f_i(x)| < \varepsilon \\ &\Rightarrow \quad |f(y) - f(x)| < \varepsilon. \end{aligned}$$

On the other hand, for $j \notin I$, because A_j is closed, there is some $\delta_j > 0$ such that

$$|y - x| < \delta_j \Rightarrow y \notin A_j.$$

Let $\delta = \min_{i \in [n]} \delta_i$. Then for all y, $|y - x| < \delta$ implies that $|y - x| < \delta_j$ for all $j \notin I$. So as the A_i cover X, $y \in A_i$ for some $i \in I$. But then $|y - x| < \delta < \delta_i$ implies that $|f(y) - f(x)| < \varepsilon$.

So f is continuous, as needed. □

1.1 Homotopies

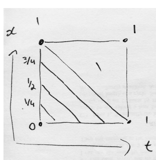
The polynomial function $p(t,x) = (1-t)x + x$ is very useful in show homotopies between functions in R^n. For example, we can show that the any continuous map $f : X \to R^n$ is homotopic to the constant map $c_1 : X \to R^n : v \to (1,1,1,\ldots,1)$ by the homotopy

$$H : I \times X \to R^n : (t,x) \mapsto (1-t)x + t(1,1,\ldots,1).$$

H is continuous as it $H = p \circ \mathrm{id} \times c_1$. All of p, id and c_1 are continuous, so $\mathrm{id} \times c_1$ is, and the composition with p is too.

As a particular case of this we get that the identity map $\mathrm{id}_I : I \to I$ is homotopic to the constant map $c_1 : \mathbb{R} \to \mathbb{R}$.

Here is a picture of H:



Just for fun lets show in another way that this map $H : I \times I \to I$ is continuous. Cover the space $I \times I$ with the lower triangle $T_l = \{(x,y) \mid x,y \geq 0, x - y \leq 1\}$ and the upper triangle T_u we get by reflecting T_l in the line $x - y = 1$. Define $f_l : T_l \to I$ by $f_l(x,y) = x - y$, and $f_u : T_u \to I$ by $f_u = c_1$. Both of these are continuous, and their union $f_u \cup f_l$ is H, so by the gluing lemma, H is continuous.

Another useful function is the the following.

Example 1.4. For $c \in I = [0,1]$ let $r_c : I \to I$ be the *reparametrisation of I* given by:

$$r_c(t) = \begin{cases} t/c & \text{if } t \leq c \\ 1 & \text{if } t \geq c \end{cases}.$$

The functions $t \to t/c$ and $t \to 1$ are both polynomial, so continuous on any subset of $\mathbb{R}$. In particular on the closed subsets $[0,c]$ and $[c,1]$ respectivley. By the Gluing Lemma, r_c is therefore continuous.

Now we can show that r_c is null-homtopic by the homotopy $H = p \circ \mathrm{id} \times r_c$. We know this is continuous, we just have to verify that it defines a map to I. This is easy.

Here is another way we can show it to be null-homotopic. Let T_l, T_u, f_l and f_u be as above. Let $H : I \times I \to I$ be defined by $H = r_c \circ f_l \cup f_u$. This is continuous. H is slightly different from the H above, but also a homotopy to c_1.

We know that both id : $I \to I$ and $r_c : I \to I$ are null-homotopic, so they are homotopy equivalent, but we can show this directly too, by the homotopy $H = \{r_{1-t+tc}\}_{t \in I}$.

Let $R_c : I^3 \to I^3$ be defined $R_c(x, y, z) = (r_c(x), r_c(y), r_c(z))$. Then this is continuous as each component function is. It takes any (x, y, z) with all entries c or greater to $(1, 1, 1)$. Similar to above, $R_{1-t+tc_{t \in I}}$ is a homotopy from $R_1 = \text{id}$ to R_c.

It is easy to find a homeomorphism F of the cube I^3 to the sphere S^2, so $F \circ R_c \circ F^{-1} : S^2 \to S^2$ is a continuous mapping $S^2 \to S^2$ which might be useful in Section 1.2, Exercise 3.

The following describes the map $S^2 \to S^2$ that Kim Se-hun was drawing in class.

Observe that the unit 2-sphere $S^2 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ can be parametrised as

$$S^2 = \{p(\theta, \phi) \mid \theta \in 2\pi \cdot I, \phi \in \pi \cdot I\}$$

where $p(\theta, \phi) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$.

Then the map

$$f_c : S^2 \to S^2 : p(\theta, \phi) \to p(\theta, \pi \cdot r_c(\phi/\pi))$$

is continuous. To show this is continuous we use the following fact. It is about what are called quotient spaces.

Proposition 1.5. *Let $f : X \to Y$ be continuous, and $q : X \to Q$ be a a continuous surjection such that f is constant on $q^{-1}(z)$ for all $z \in Q$. . Then $f/q : Q \to Y : x \to f \circ q^{-1}$ is well-defined and continuous.*

Now $F_C : 2\pi \cdot I \times \pi \cdot I \to S^2 : (\theta, \phi) \to p(\theta, \pi \cdot r_c(\phi/\pi))$ is continuous as its component maps are compositions of continuous maps.

Moreover $f_c = F_C/p$, so by the proposition f_c is continuous.