Linear Algebra Midterm Solutions

1. Answer 'T(rue)' or 'F(alse)'.

   (a) If columns 1 and 3 of B are the same, then columns 1 and 3 of AB are.

   (b) If rows 1 and 3 of B are the same, then rows 1 and 3 of AB are.

   (c) If $AB = AC$ then $B = C$.

   (d) The set of vectors $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ with $b_1 = 1$ is a subspace of $\mathbb{R}^3$.

   (e) If A is invertible, then the nullspace is empty.

   (f) Any three independent vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$.

   (g) Given a matrix A, if we reduce the augmented matrix $[A \mid I]$ by Gaussian Elimination to $[A' \mid B]$, then $BA = I$.

Solution

	T or F	Comment/Reason
a	T	
b	F	
c	F	True if A is invertible. But it isn't.
d	F	$(1, 0, 0)$ is in the set but $2(1, 0, 0)$ isn't.
e	F	The nullspace always contains the zero vector.
f	T	
g	F	$BA = A'$.

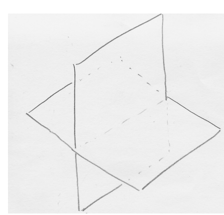
2. Find a point with $z = 3$ on the intersection of the planes $x + y + 3z = 6$ and $x - y + z = 4$.

Solution

Putting $z = 3$ into the equations we get $x+y = -3$ and $x-y = 1$. Solving this yields $2x = -2$ so $x = -1$ and $y = -2$. So the point is $(x, y, z) = (-1, -2, 3)$.

3. The picture represents the matrix equation

$$Ax = b.$$



What are the dimensions of A?

How many pivot rows does it have?

Solution

The objects are planes in $\mathbb{R}^3$ so represent equations in $\mathbb{R}^3$. So A has three columns. There are two of them, so there are two rows. So A is 2×3. With two rows there can be at most 2 pivots, and as the two planes are independent, there are exactly 2 pivots.

4. Find a, b and c where

$$\begin{bmatrix} a & 2 \\ 1 & 1 \\ b & 4 \end{bmatrix} \begin{bmatrix} c \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 9 \end{bmatrix}.$$

Solution

From the second row we get that $c = -1$. So $a = 2$ and $b = 3$.

5. (a) Find an LU decomposition of the matrix

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 1 \end{bmatrix}.$$

(b) Find another one.

Solution

Using Gauss Jordan elimination:

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 0 & -\frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right] =: [U \mid L^{-1}]$$

we have that $L^{-1}A = U$. As L^{-1} is the row matrix that takes half the first row from the second, so its inverse is the matrix that adds half the first row to the second:

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}.$$

So

$$\begin{bmatrix} 4 & 7 \\ 2 & 1 \end{bmatrix} = A = LU = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 0 & -\frac{5}{2} \end{bmatrix}$$

is an LU decomposition of A.

Pulling out the diagonal $D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 0 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 0 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & \frac{7}{2} \\ 0 & -\frac{5}{2} \end{bmatrix}$$

We get that

$$\begin{bmatrix} 4 & 7 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & \frac{7}{2} \\ 0 & -\frac{5}{2} \end{bmatrix}$$

is another LU decomposition.

6. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$.

Solution

$$A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

7. Where A is the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ solve $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Solution

Eliminate the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

8. Assume that a matrix equation $Mx = b$ has solutions $(1, 2, 3)$ and $(2, 2, 2)$, what is another solution? (Note, b is not necessarily the zero vector.)

Solution

Another one is $\frac{1}{2}(1, 2, 3) + \frac{1}{2}(2, 2, 2) = (3/2, 2, 5/2)$. Indeed $M(\frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}) = \frac{1}{2} M \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2} M \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{2}b + \frac{1}{2}b = b$

9. The vectors w_1, w_2, and w_3 are independent. Show whether or not the vectors $v_1 = w_1 + w_1$, $v_2 = w_1 + w_2$ and $v_3 = w_1 + w_3$ are are independent.

Solution

Method 1: If $a_1v_1 + a_2v_2 + a_3v_3 = 0$ then substiting in the values of the v_i we get $(2a_1 + a_2 + a_3)w_1 + a_2w_2 + a_3w_3 = 0$. As the w_i are independent, this implies that $(2a_1 + a_2 + a_3), a_2$, and a_3 are 0. So $a_1 = a_2 = a_3 = 0$. This shows that the v_i are independent.

Method 2: Show that the v_i generate the w_i: $w_1 = v_1/2$, $w_2 = v_2 - v_1/2$, and $w_3 = v_3 - v_1/2$, so generate the same space as the v_i. Being 3 vectors generating a 3 dimensional space, they are a basis.

Method 3: Where W is the matrix with i^{th} row w_i and V is the matrix with i^{th} row v_i we have that

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} W = V.$$

As this matrix is invertible, the rowspaces of W and V are the same dimension, 3 (as the w_i are independent), so the v_i are independent.

10. Find bases for the four fundamental subspaces of A where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution

The given representation comes from a Gauss Jordan elimination. Rowspace: $\{(1,2,3,4),(0,1,2,3),(0,0,1,2)\}$
Colspace: Is $\mathbb{R}^3$ so $\{(1,0,0),(0,1,0),(0,0,1)\}$ or maybe you put $\{(1,6,9),(0,1,8),(0,0,1)\}$.
Nullspace: Finishing the elimination:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to $Ax = 0$ is $(w, x, y, z) = z(0, 1, -2, 1)$ so the basis of the nullspace is $\{(0, 1, -2, 1)\}$.

Left Nullspace: Has dimension $m - r = 3 - 3$ so $\{(0, 0, 0)\}$ or 'empty set' or 'there is no basis' are all okay.

11. Construct a matrix whose nullspace is generated by the vectors $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.

Solution

To get a nullspace of dimension 2, the matrix A should eliminate to something with two pivots. Lets just take A as the eliminated matrix:

$$A = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \end{bmatrix}$$

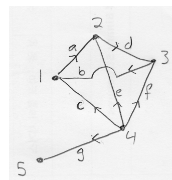
So that $(2,2,1,0)$ is in the null-space, the third column must be

$$A = \begin{bmatrix} 1 & 0 & -2 & \\ 0 & 1 & -2 & \end{bmatrix}$$

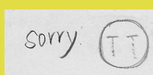
and so that $(3,1,0,1)$ is in the nullspace the fourth column must then be

$$A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

12. (a) Give the adjacency matrix of the pictured graph. (You can omit '0's. Please label columns and rows.)
    (b) What is the dimension of the cycle space?
    (c) Give a basis of the cycle space.



Solution



I was thinking of something else. We studied the *incidence matrix*, not the *adjacency matrix*. This is another matrix used to describe a graph. Luckily, all but one of you gave the indicence matrix. Also, as many of you suggested, we studied it for oriented graphs, not graphs. So I should have put an orientation on the graph. Again, most of you did this yourselves. The figure now has an orientation. The solution below uses this orientation.

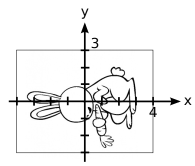
Adjacency Matrix:

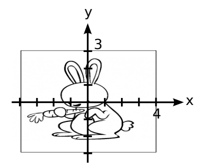
	1	2	3	4	5
a	-1	1			
b	1		-1		
c	1			-1	
d		-1	1		
e		1		-1	
f			1	-1	
g				-1	1

The graph is connected, so the space has dimension $m-(n-1) = 7-4 = \mathbf{3}$. The cycles $(a,-e,c)$, $(e,d,-f)$ and (a,d,b) are a basis of the cycle space.

Indeed, they are represented (with respect to some orientation of the graph) by the column vectors $(1,0,1,0,-1,0,0)$, $(0,0,0,1,1,-1,0)$ and $(1,1,0,1,0,0,0)$ respectively. These are clearly independent, so make up a basis.
One can also see this by taking the spanning tree of all edges incident to 4, and observing that adding the edges a, d and b make these cycles respectively.)

13. A matrix A does the transformation below. (Every point $(x,y)^T$ in the picture on the left, has been moved to $A(x,y)^T$ in the picture on the right.) Give the matrix A.

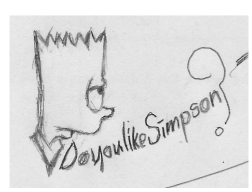




Solution

The picture has been rotated by $-\pi/2$ radians, then scaled in the x direction by a factor of $4/3$ and in the y direction by a factor of $3/4$. So

$$A = \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{4}{3} \\ -\frac{3}{4} & 0 \end{bmatrix}.$$



Some artwork from the test.