Set Theory 2018 Midterm Solutions

1. Use a (simplified) truth table to prove that $(p \to q) \Rightarrow ((p \wedge r) \to (q \wedge r))$.

Solution

Here is a simplified truth table:

p	$\to$	q	$\to$	p	$\wedge$	r	$\to$	q	$\wedge$	r
0	1	0	1	0	0	0	1	0	0	0
0	1	0	1	0	0	1	1	0	0	1
0	1	1	1	0	0	0	1	1	0	0
0	1	1	1	0	0	1	1	1	1	1
1	0	0	1	1	0	0	1	0	0	0
1	0	0	1	1	1	1	0	0	0	1
1	1	1	1	1	0	0	1	1	0	0
1	1	1	1	1	1	1	1	1	1	1
1	2	1	4	1	2	1	3	1	2	1

The final column (labelled on the bottom with a 4) is all 1s. So this is a tautology.

2. Use deductive reasoning to prove the following:

$$(p \to r) \wedge (q \to r) \Rightarrow (p \vee q) \to r.$$

Solution

$$\begin{aligned} (p \to r) \wedge (q \to r) &\Rightarrow (p \vee q) \to (r \vee r) && \text{CD} \\ &\iff (p \wedge q) \to r && \text{Idem.} \end{aligned}$$

3. Prove: $\sim [(\forall x)(\sim q(x))] \equiv (\exists x)(q(x))$.

Solution

$$\begin{aligned} \sim [(\forall x)(\sim q(x))] &\equiv (\exists x)(\sim\sim q(x)) && \text{QN} \\ &\equiv (\exists x)(q(x)) && \text{DN} \end{aligned}$$

4. Prove or disprove the following identity:

$$(p \to q) \wedge (q \to r) \iff (p \vee q) \to r.$$

Solution

Setting $p = r = 1$ and $q = 0$ we see that

$$(p \to q) \wedge (q \to r) = (1 \to 0) \wedge (0 \to 1) = 0 \wedge 1 = 0$$

while

$$(p \vee q) \to r = (1 \vee 0) \to 1 = 0 \to 1 = 1.$$

This is a counter-example showing that the identity is false.

5. Prove the following argument.

$$\begin{aligned} A \wedge B &\to C \\ (A \to C) &\to D \\ \sim B &\vee E \qquad / \therefore B \to D \wedge E \end{aligned}$$

Solution

This is a bit hard, so you should look at it and develop a strategy before you start. Your goal is $B \to D \wedge E$, and immediatly you see that the third premise gives you $B \to E$, so it is enough to get $B \to D$. Clearly the third premise will not help in this, so we look at the first two. The second $(A \to C) \to D$ seems to give us D, if we can get $A \to C$. So can we get this from the first premise ? Remember the Rule of Exportation? It was exercise 1.5.7, we did it in class. Okay, this was all before you started writing. Now you have to write this argument down.

1)	$(B \wedge A) \to C$	Given (and commutativity)
2)	$B \to (A \to C)$	1, Exportation. (Or ...
1.1)	$\sim(B \wedge A) \vee C$	1 and Def of $\to$
1.2)	$\sim B \vee (\sim A \vee C)$	1.1 and DeM. and Dist.
1.3)	$\sim B \vee (A \to C)$	1.2 and Def of $\to$
2)	$B \to (A \to C)$	1.3 and Def of $\to$)
3)	$(A \to C) \to D$	Given
4)	$B \to D$	2,3 and Transitivity (Syllogism).
5)	$\sim B \vee E$	Given
6)	$B \to E$	5, Def of $\to$.
7)	$(B \wedge B) \to (D \wedge E)$	4,6, CD.
8)	$B \to (D \wedge E)$	7, Simp.

6. Prove that for all natural numbers n,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

Solution

We prove this by induction. For the base case $n = 1$, we have $\frac{1}{1\cdot 2} = \frac{1}{1+1}$. Now, assume $n > 1$ and that the statement holds for all values less than n. Then by the induction hypothesis we have that

$$\begin{aligned}\frac{1}{1\cdot 2} + \cdots + \frac{1}{n\cdot(n+1)} &= \frac{n-1}{n} + \frac{1}{n\cdot(n+1)} = \frac{(n-1)(n+1)+1}{n(n+1)} \\ &= \frac{n^2-1+1}{n(n+1)} = \frac{n}{n+1}\end{aligned}$$

as needed for the result to follow by induction.

7. Prove or give a counterexample: If $x \in A$ and $A \in B$ then $x \in B$.

Solution

This is not true. Indeed let $A = \{1\}$ and $B = \{A\}$, then $1 \in A$ and $A \in B$, but $1 \notin B$, as B contains only the one element A.

8. Where $A = \{a, b, c, d\}$, how many elements are there in the set

$$\{S \in \mathscr{P}(A) \mid (b \in S) \vee (d \notin S)\}?$$

Solution

Twelve: the set is the union

$$\{S \in \mathscr{P}(A) \mid b \in S\} \cup \{S \in \mathscr{P}(A) \mid d \notin S\}.$$

Both of these sets have $2^3 = 8$ elements, and there are four elements: $\{b\}, \{a, b\}, \{b, c\}$, and $\{a, b, c\}$, in both of them. So the set contains $8+8-4 = 12$ elements.

9. Prove: $(A \cup B = A \cap B) \iff A = B$.

Solution

First, assume that $A \cup B = A \cap B$. We show $A = B$ by showing $A \subseteq B$ and $B \subseteq A$. Indeed,

$$\begin{aligned} A &\subseteq A \cup B && \text{Add: } (x \in A) \Rightarrow (x \in A) \vee (x \in B) \\ &= A \cap B && \text{By Assumption} \\ &\subseteq B && \text{Simp: } (x \in A) \wedge (x \in B) \Rightarrow (x \in B), \end{aligned}$$

and the same argument gives

$$B \subseteq A \cup B = A \cap B \subseteq A,$$

so $A = B$. Thus we have shown $A \cup B = A \cap B \Rightarrow A = B$.
On the other hand, assume that $A = B$. Then

$$\begin{aligned} A \cup B &= B \cup B && A = B \\ &= B && \text{Ident.} \\ &= B \cap B && \text{Ident.} \\ & A \cap B && A = B. \end{aligned}$$

This shows $A \cup B = A \cap B \Leftarrow A = B$. So $A \cup B = A \cap B \iff A = B$

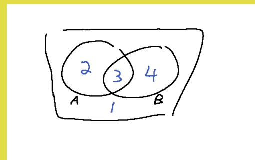
10. Prove: $A - B = A - (B \cap A)$.

Solution

$$\begin{aligned} A - (B \cap A) &= A \cap (B \cap A)' && \text{Def of } - \\ &= A \cap (B' \cup A') && \text{DeMorg.} \\ &= (A \cap B') \cup (A \cap A') && \text{Dist.} \\ &= (A \cap B') \cup \emptyset && \text{Compl.} \\ &= A \cap B' && \text{Ident.} \\ &= A - B && \text{Def of } - \end{aligned}$$

11. ‘Prove’ the previous question with a Venn Diagram.

The Venn diagram shows the two sets A and B with a universe of four representative elements; one for each possible pattern of inclusions in A and B.



Where $A = \{2, 3\}$ and $B = \{3, 4\}$ we get

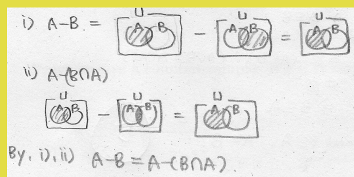
$$A - B = \{2, 3\} - \{3, 4\} = \{2\}$$

and

$$A - (B \cap A) = \{2, 3\} - (\{3, 4\} \cap \{2, 3\}) = \{2, 3\} - \{3\} = \{2\}.$$

As both $A - B$ and $A - (B \cap A)$ consist of the same set $\{2\}$ of representitive elements, they are the same set.

Alternately, here is very nice 'proof' that one of you gave.



12. Find (without proof) the intersection and union of the family of open intervals:

$$\{(0, 1/n) \mid n \in \mathbb{N}\}.$$

The intersection is $\bigcap_{n=1}^{\infty}(0, 1/n) = \emptyset$, union is $\bigcup_{n=1}^{\infty}(0, 1/n) = (0, 1)$.

13. Prove: $\left(\bigcap_{\gamma \in \Gamma} A_\gamma\right)' = \bigcup_{\gamma \in \Gamma} A'_\gamma$.

Solution

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\begin{aligned}
x \in \Big(\bigcap_{\gamma\in\Gamma} A_\gamma\Big)' &\iff \sim\Big(x \in \bigcap_{\gamma\in\Gamma} A_\gamma\Big) && \text{Def of }' \\
&\iff \sim(\forall\gamma \in \Gamma)(x \in A_\gamma) && \text{Def of } \bigcap \\
&\iff (\exists\gamma \in \Gamma)(x \notin A_\gamma) && \text{QN} \\
&\iff (\exists\gamma \in \Gamma)(x \in A'_\gamma) && \text{Def of }' \\
&\iff \bigcup_{\gamma\in\Gamma} A'_\gamma && \text{Def of } \bigcup.
\end{aligned}
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