Graph Theory 2020 Midterm Test

- Do not talk to other students about the exam.
- For full marks, show work, give reasons, and write clearly.
- Clearly $\boxed{\text{box}}$ your final answer if there is one.
- **Do four of the following six questions.** Be clear about which ones you are not doing. If you do more, you will only get credit for the first four you do.
- Return to mhsiggersknu.ac.kr by **3pm on Thursday May 14.** (Not Tuesday, as originally planned.)

1. (a) Decide if the sequence $(6, 5, 5, 3, 2, 2, 2, 1)$ is graphic. If it isn't, change it so that it is.

   (b) A pair of non-increasing sequences $(a_1, a_2, \ldots, a_m)$ and $(b_1, b_2, \ldots b_n)$ of positive integers is *bigraphic* if there is a bipartite graph with partite sets $U = (u_1, \ldots, u_m)$ and $V = (v_1, \ldots, v_n)$ such that $\deg(u_i) = a_i$ for all $i \in [m]$ and $\deg(v_i) = b_i$ for all $j \in [n]$. Is the pair

   $$(4, 3, 2, 2, 1, 1) \text{ and } (6, 4, 2, 2)$$

   bigraphic? If it isn't, change it so that it is.

   (c) Prove or disprove: a pair of non-increasing positive integer sequences is bigraphic if and only if the sums of degrees in the two sequences are the same.

2. A graph G is *self-complementary* if $G = \overline{G}$.

   (a) Give an example of a self-complementary graph on 5 vertices.

   (b) How many edges does a self-complementary graph on n vertices have?

   (c) Is there a self-complementary graph on n vertices for all $n \geq 5$?

   (d) Show that the chromatic number of a self-complemenary graph on n is at least $\sqrt{n}$.

3. Show that any odd circuit in a graph has an odd cycle as a subgraph.

4. The *n-wheel* W_n is the graph we get from the cycle C_n by adding a new vertex adjacent to all other vertices. More generally, W_n^m is the graph we get from C_n and by adding a new independent set M of m vertices, and adding all edges from C_n to M.

   (a) How many labelled hamilton cycles are there in W_n?

   (b) How many labelled hamilton cycles are there in W_n^2?

   (c) How many in W_n^m? **or** Write a python program that counts this for inputted m and n.

5. A graph is a *tree-plus-three* if we can get it from a tree T by adding 3 edges.

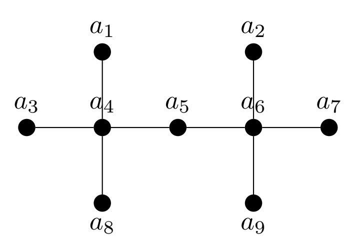
   (a) Draw a tree-plus-three on 10 vertices.

   (b) What is the minimum number of cycles in a tree-plus-three?

   (c) What is the maximum number of cycles in a tree-plus-three?

6. Take a random permutation $(a_1, a_2, \ldots, a_9)$ of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$.

   (a) Give the Pruffer decoding of the following tree:



   (b) Give the Pruffer encoding of the following sequence: $(a_3, a_6, a_1, a_1, a_5, a_3, a_1)$.

   (c) Show that the operations of encoding and decoding are inverse operations.