Linear Algebra 2020 Midterm Solutions

1. Answer 'T(rue)' or 'F(alse)'.

   (a) If $AB = AC$ then $B = C$. [F– True if A is invertible.]

   (b) The set of vectors $b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ with $b_1 - b_2 + 3b_3 = 0$ is a subspace of $\mathbb{R}^3$. [T]

   (c) The matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has rank 2. [T: Two pivots]

   (d) There are n different $n \times n$ permutation matrices. [F: There are $n!$ of them.]

   (e) If $(3, 4, 5)$ and $(1, 2, 0)$ are both solutions to $Mx = b$ then M is singular. [T]

   (f) The nullspace of the incidence matrix M of a graph G always has dimension 1. [F: Only if G is connected.]

   (g) If $A^T = -2A$ then the rowspace of A equals the column space of A. [T]

2. Write the following system of equations as a matrix equation, and solve it $2u + v = 5$, $3v - w = 3$, $u + 6w = 7$, $2u - w + z = 12$.

Solution

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 1 & 0 & 6 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 7 \\ 12 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ w \\ v \\ u \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

which we get by re-arranging columns and rows, to make elimination easier: We then solve via Gauss-Jordan':

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 12 \\ 0 & -1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 6 & 0 & 1 & 7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 12 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & -35 & -65 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 12 \\ 0 & 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 13/7 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{84+6-2(13)}{7} = 64/7 \\ 0 & 1 & 0 & 0 & \frac{-21+3(9)}{7} = 6/7 \\ 0 & 0 & 1 & 0 & \frac{35-2(13)}{7} = 9/7 \\ 0 & 0 & 0 & 1 & 13/7 \end{array}\right]$$

So $(u, v, w, z) = \frac{1}{7}(13, 9, 6, 64)$

3. Find a 3×3 matrix A with nine distinct entries, but in which row 2 and 3 become zero in elimination. How many solutions does $Ax = b$ have when $b = (1, 10, 100)$ and how many when $b = (0, 0, 0)$.

Solution

There are plenty examples of such A. They are of the form $\begin{bmatrix} x & y & z \\ ax & ay & az \\ bx & by & bz \end{bmatrix}$ where x, y and z are distinct, and you just have to be a little careful with a and b. When $b = (1, 10, 100)$, the system $Ax = b$ will probably have no solutions, unless you had $a = 10$ and $b = 100$, in which case the solution you have a plane of solutions: for any x_1, and x_2 there is x_3 so that $x = (x_1, x_2, x_3)$ is a solution. When $b = (0, 0, 0)$ there is a plane of solution. (This is a question from Section 1.4. Using language of Section 2.4, the matrix has rank 1 so its null-space (the solution space of $Ax = 0$) has dimension 2 . If $(1, 10, 100)$ is in the column space then the general solution of $Ax = b$ is a plane, otherwise, it is empty.)

4. Write each of the following as a single matrix or number.

$$a) \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^4 \qquad b) \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

(Hint: You shouldn't have to do these the long way. Think of what they do as elementary matrices.)

Solution

a) This matrix performs the row operation $R_2 = R_2 - 3R_1$. This to the power 4 does this 4 times so is $R_2 = R_2 - 12R_1$: $\boxed{\begin{bmatrix} 1 & 0 \\ -12 & 0 \end{bmatrix}}$. b) The inverse of the row operation $R_2 = R_2 + 2R_1$ is $R_2 = R_2 - 2R_1$: $\boxed{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$.

5. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$.

Solution

I expect you to do Guassian Elimination here. You should have got: $A^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$

6. Solve as two triangular systems (without finding LU) the equation

$$LUx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

Solution

Solving first

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} c = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

we get $c = (2, -2, 0)$. And then solving

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

we get $(x, y, z) = (5, -2, 0)$.

7. Where A is the matrix from Question 5, solve

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

8. Find bases for the four fundamental subspaces of A where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Solution

The given representation comes from a Gauss Jordan elimination. Rowspace: $\{(1,2,3,4),(0,1,2,3),(0,0,1,2)\}$
Colspace: Is $\mathbb{R}^3$ so $\{(1,0,0),(0,1,0),(0,0,1)\}$ or maybe you put $\{(1,6,9),(0,1,8),(0,0,1)\}$.
Nullspace: Finishing the elimination:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution to $Ax = 0$ is $(w,x,y,z) = z(0,1,-2,1)$ so the basis of the nullspace is $\{(0,1,-2,1)\}$.

Left Nullspace: Has dimension $m - r = 3 - 3$ so $\{(0,0,0)\}$ or 'empty set' or 'there is no basis' are all okay.

9. What 3×3 matrix represents the linear transformation that rotates the xy-plane by $90°$ but leaves the z-axis alone?

Solution

It should take e_1 to $(0,1,0)$, e_2 to $(-1,0,0)$ and e_3 to e_3. So it is $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

10. Which of these transformation **are** linear? Where $v = (v_1, v_2)$,

(a) $T(v) = (v_2, v_2)$.

(b) $T(v) = (1, v_2)$.

(c) $T(v) = (-v_2, v_1 + v_2)$.

(d) $T(v) = (v_1^2, -(v_2^2))$.

Solution

(a) and (c) are, they are multplication by the matrices $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ respectively.

(b) isn't: $T(1,1) = (1,1) \neq (1,1) + (1,0) = T(0,1) + T(1,0)$.

(d) isn't: $T(2,0) = (4,0) \neq (1,0) + (1,0) = T(1,0) + T(1,0)$.