A REFLEXIVE DIGRAPH THAT IS RETRACTION TRIVIAL BUT NOT ENDOMORPHISM TRIVIAL

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Abstract. In this note we give an example of a reflexive digraph that has no non-trivial retractions, but does have non-trivial endomorphisms.

1. Introduction

A directed graph, or *digraph*, is reflexive if every vertex has an arc to itself. Recall that a *homomorphism* $\phi: G \to H$ from a digraph G to another digraph H, is an arc preserving vertex map. An *endomorphism* is a homomorphism $\phi: G \to G$ of a digraph to itself, while a *retraction* $\phi: H \to H$ is an endomorphism such that for any vertex v in its image, $\phi(v) = v$. An endomorphism is *constant* if its image contains a single vertex, and is *trivial* if it is constant or an automorphism. The image of a retraction of H is a *retract* of H. A digraph is *endo-trivial* if all of its endomorphisms are trivial, it is *retract-trivial* if all of its retractions are trivial. Clearly endo-trivial graphs are retract-trivial.

The authors of [1], where the notion of endo-triviality is defined, asked via a personal communication if the converse is true: 'Are all retract-trivial digraphs endo-trivial?' It seemed that this should not be true, but examples of endo-trivial digraphs that are not retract-trivial proved unexpectedly hard to find. After several days, I finally constructed such a digraph– an ugly beast of a thing. Through several frustrating attempts at writing up a description of the graph, it got neater and neater, and smaller and smaller, and I began to fear that if I did not finally commit to a description, the graph might disappear completely. In this note I allay this fear by presenting a final version of the digraph. One wonders how much smaller an example there can be.

I refer the reader to [1] for a deeper motivation.

2. Definitions and initial observations

Recall that a digraph is *strongly connected* if for any two vertices u and v there is a directed path from u to v. We abuse notation slightly and use '$u \to v$' to both mean that (u, v) is an arc, and to refer to that arc. The following observations are easy.

Fact 2.1. *Any retract-trivial reflexive digraph H is oriented and strongly connected.*

Proof. If H has a symmetric edge $u \to v \to u$, then H retracts onto this edge. If H is not strongly connected, then there is a strong component U with no arcs from any other strong component. Letting u be a vertex of U with an arc to v in another

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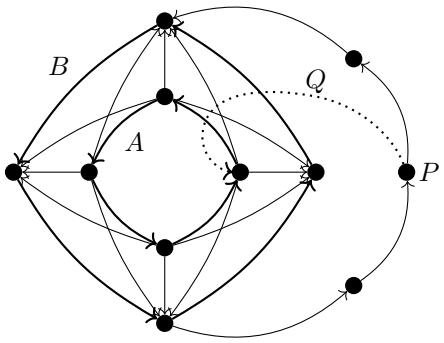

FIGURE 1. The digraph G

strong component, we can retract H to the reflexive arc $u \to v$ by mapping all of U to u and everything else to v. □

In light of this fact, all digraphs in the remainer of the note are reflexive oriented digraphs. Oriented graphs have no cycles of girth 2 and cycles of girth one we will call *loops*, so by a *cycle*, we mean a directed cycle of girth at least 3. Let C_d be the directed cycle of girth d with arcs $0 \to 1 \to \cdots \to d-1 \to 0$.

3. The digraph G

Let G be the digraph, depicted in the figure, which is constructed as follows. Starting with disjoint copies A and B of C_4, add all arcs from A to B except a matching. Add a path P of length 4 between antipodal vertices of B, and add a path Q of length 5 from the middle vertex of P to the first vertex of A. We point out a few easily verifiable features of G.

Fact 3.1. *Where G' is the subgraph of G induced by the vertices of $A \cup B \cup P$, the following are true.*

- *(i) G' had two strong components: A and $B \cup P$.*
- *(ii) G' has three cycles: A and B, having girth 4, and one more in $B \cup P$ having girth 6.*
- *(iii) The only other cycles in G contain Q and at least four vertices of G'.*
- *(iv) G contains no vertex that has arc to every vertex of a cycle, and no vertex that has arc from every edge of a cycle.*

With these observations we prove the following.

Theorem 3.2. *The digraph G defined above is retract-trivial but not endo-trival.*

Proof. To see that G is not endo-trivial we observe that there is a homomorphism ϕ of G to C_4. Indeed let ϕ map all of A to the vertex 0 of C_4 and all of B to 1. This maps the end points of P to 1, and we can map the remaining vertices of P surjectively around C_4. The initial vertex of Q is thus mapped to 3 and its final vertex to 0, As Q has length 5, the rest of it can clearly be mapped to C_4. Composing this with an isomorphism of C_4 to A we have a non-trivial (non-retraction) endomorphism of G to the subgraph A.

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Now, let r be a retraction of G and $R = r(G)$. We will show that r is trivial. We start with a couple of simple observations, the first two coming from the easy observation that the homomorphic image of a strongly connected digraph is strongly connected, and the last coming from Fact 3.1 (iv) and the observation that every vertex of B has an arc from a vertex of A, and every vertex of A has an arc to a vertex of B.

(i) R is strongly connected.
(ii) For any cycle C of G, $r(C)$ is a cycle or a loop.
(iii) If $r(A)$ is a loop then $r(B)$ is, and vice-versa.

By (ii), $r(A)$ is a cycle or a loop, which breaks our proof into two cases.

Assume first that $r(A)$ is a loop. By (iii), $r(B)$ is also a loop. As the endpoints of P are both in B, we thus have that P maps to the loop $r(B)$ or to a cycle of length at most four. However, as r maps A and B to loops, R contains no cycle not containing Q, so certainly no cycle of length at most 4. Thus $r(P) = r(B)$; from which we conclude that R contains at most two vertices of G'. If $r(A) \neq r(B)$ then $r(A) \to r(B)$ and as R is strongly connected, it must contain some path from $r(B)$ to $r(A)$, which by Fact 3.1 (iii) must contain Q. But this contains at least 4 vertices of G', contradicting the fact that R contains at most two vertices of G'. Thus $r(A) = r(B)$, giving us that $r(G')$, and so $r(G)$, is a loop. This shows that r is trivial, as needed.

We may, therefore, assume that $r(A)$ is a cycle. In particular, by Fact 3.1 (ii) we have that $r(A)$ is A or B. If $r(A) = B$ then B is a subgraph of R and so r retracts A onto B. This is impossible though, as every vertex of A has three out-neighbours in B but there is no vertex in B that has such out-neighbours. So $r(A) = A$. The same argument with 'in-neighbours' in place of 'out-neighbours' tells us that $r(B)$ cannot be A, and as, by (iii), it must map to a cycle of girth 4, we have $r(B) = B$. R therefore contains vertices of two strong components of G', so as R is strongly connected, it must also contain Q. This forces $r(p) = p$ for every vertex in P, giving us that $R = G$. Again, r is trivial, as needed.

□

References

[1] B. Larose, B. Martin, and D. Paulusma. Surjective h-colouring over reflexive digraphs. *TOCT*, 11:3:1–3:21, 2019.

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