RELATIONAL STRUCTURES ADMITTING CERTAIN 6-ARY TAYLOR POLYMORPHISMS ARE NP-COMPLETE TARGETS

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Abstract. Here we give the combinatorial argument that is at the heart of my paper "A Strong Mal'cev Condition for Locally Finite Varieties omitting Type 1".

1. Introduction

In [6], I showed that the problem of deciding if a core relational structure $\mathcal{H}$ is one of those that is conjectured in [1] to yield a polynomial time solvable problem $\mathrm{CSP}(\mathcal{H})$, is itself an NP problem. As the result had more interesting consequences in Universal Algebra, I wrote the proof from that point of view. Here I give my original proof, which is purely combinatorial. The proof is an example of the fibre construction developed in [5], [3], and [4]. We prove the following.

Theorem 1.1. *If a core relational structure has no idempotent polymorphisms ϕ such that the following identity holds for all pairs (x, y) of vertices in $V(\mathcal{H})$, then* $\mathrm{CSP}(\mathcal{H})$ *is NP-complete.*

$$\phi(x,x,x,x,y,y) = \phi(x,y,x,y,x,x) \text{ and } \phi(y,y,x,x,x,x) = \phi(x,x,y,x,y,x). \quad (1)$$

A polymorphism ϕ of $\mathcal{H}$ is *idempotent* if $\phi(v, v, \ldots, v) = v$ for all vertices v in $\mathcal{H}$. This and all other basic CSP definitions can be found on [4].

We use only the fact, which follows from the H-colouring Dichotomy of Hell and Nešetřil [2] (a paper which apparently I must cite in everything I write) that $\mathrm{CSP}(T)$ is NP-complete for any simple graph T containing a triangle.

Proof. Let $\mathcal{H}$ be a core relational structure having no idempotent 6-ary polymorphism satisfying identities (1). It is well known, see [4], that a core $\mathcal{H}$ is NP-complete if and only if its idempotent reduct is, thus we will assume that the only polymorphisms of $\mathcal{H}$ are idempotent.

We give a polynomial time construction that provides, for any instance G of $\mathrm{CSP}(T)$, an instance G^* of $\mathrm{CSP}(\mathcal{H})$ such that

$$G \to T \iff G^* \to \mathcal{H}.$$

This encodes $\mathrm{CSP}(T)$, which is NP-complete, in $\mathrm{CSP}(\mathcal{H})$, so $\mathrm{CSP}(\mathcal{H})$ is also NP-complete.

It the proof we will use several copies of some ordered vertex set W^*, and we will consider a function defined on one of these copies, to be defined on another copy. So if I have copies $W^1 = (v_1^1, \ldots v_w^1)$ and $W^2 = (v_1^2, \ldots, v_w^2)$ of $W^* = (v_1, \ldots, v_w)$, and define the function f on W^1, then say some other function ϕ is equal to f on W^2, then I mean $\phi(v_i^2) = f(v_i^1)$ for all i. If we say we identify two such copies, say W^1 and W^2 vertexwise, we mean we identify v_i^1 and v_i^2 for all i.

$$W^1 = \Big(\begin{array}{llllll} (v_1, & v_1, & v_1, & v_1, & v_2, & v_2), \\ (v_2, & v_2, & v_1, & v_1, & v_1, & v_1), \\ (v_2, & v_2, & v_2, & v_2, & v_1, & v_1), \\ (v_1, & v_1, & v_2, & v_2, & v_2, & v_2), \\ \hline (v_1, & v_1, & v_1, & v_1, & v_3, & v_3), \\ (v_3, & v_3, & v_1, & v_1, & v_1, & v_1), \\ (v_3, & v_3, & v_3, & v_3, & v_1, & v_1), \\ (v_1, & v_1, & v_3, & v_3, & v_3, & v_3), \\ \hline & & & \dots & & \\ \hline (v_{n-1}, & v_{n-1}, & v_{n-1}, & v_{n-1}, & v_n, & v_n), \\ (v_n, & v_n, & v_{n-1}, & v_{n-1}, & v_{n-1}, & v_{n-1}) \\ (v_n, & v_n, & v_n, & v_n, & v_{n-1}, & v_{n-1}), \\ (v_{n-1}, & v_{n-1}, & v_n, & v_n, & v_n, & v_n) \end{array} \Big)$$

$$W^2 = \Big(\begin{array}{llllll} (v_1, & v_2, & v_1, & v_2, & v_1, & v_1), \\ (v_1, & v_1, & v_2, & v_1, & v_2, & v_1), \\ (v_2, & v_1, & v_2, & v_1, & v_2, & v_2), \\ (v_2, & v_2, & v_1, & v_2, & v_1, & v_2), \\ \hline (v_1, & v_3, & v_1, & v_3, & v_1, & v_1), \\ (v_1, & v_1, & v_3, & v_1, & v_3, & v_1), \\ (v_3, & v_1, & v_3, & v_1, & v_3, & v_3), \\ (v_3, & v_3, & v_1, & v_3, & v_1, & v_3), \\ \hline & & & \dots & & \\ \hline (v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_{n-1}), \\ (v_{n-1}, & v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_{n-1}) \\ (v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_n, & v_n), \\ (v_n, & v_n, & v_{n-1}, & v_n, & v_{n-1}, & v_n) \end{array} \Big).$$

FIGURE 1. Subsets W^1 and W^2 of $M = \mathcal{H}^6$: In each of W^1 and W^2, the same four 6-tuples are repeated for each pair of vertices of $\mathcal{H}$.

First we build an edge-replacement gadget M.

The Gadget M. Let $V(\mathcal{H}) = \{v_1, \dots, v_n\}$. Let $M = \mathcal{H}^6$ and let W^1 and $W^2 \subset V(M)$ be the ordered sets of $4 \cdot \binom{n}{2}$ 6-tuples each shown in Figure 1.

The Graph T. Let W^* be an ordered set of $4 \cdot \binom{n}{2}$ vertices. We consider both W^1 and W^2 as copies of W^*, with vertices in the orders shown in the figure. , and any function Let T be the digraph with vertex set

$$V(T) = \{\psi|_{W^1} \mid \psi : H^6 \to H \text{ is a homomorphism}\}$$

and arc set defined $f \to g$ if there is a homomorphism $\psi : H^6 \to H$ such that $\psi|_{W^1} = f$ and $\psi|_{W^2} = g$.

Since $\sigma : (x_1, x_2, x_3, x_4, x_5, x_6) \to (x_3, x_5, x_1, x_6, x_2, x_4)$ is an automorphism of $M = \mathcal{H}^6$ that maps W^1 to W^2 and W^2 to W^1, $E(T)$ is well defined and symmetric.

Observe also that T is irreflexive (loopless). Indeed any homomorphism $\phi : M \to \mathcal{H}$ is an idempotent 6-ary polymorphism of $\mathcal{H}$, so the assumption of the theorem gives us that there is some choice of $x = v_i$ and $y = v_j$ that does not satisfy the identities (1). Thus $\psi|_{W^1}$ and $\psi|_{W^2}$ are different functions, so T has no loops.

Finally, observe that T contains a triangle. Indeed, the projections of $M = \mathcal{H}^6$ restrict on W^1 and on W^2 to only three different functions (vertices of T) and that each of the six projections puts a different arc in $E(T)$ between these functions.

The Fibre Construction $G \mapsto G^*$. Given a graph G, arbitrarily orient its edges, and let be G^* defined as follows.

(i) For each vertex v of G let W^v be a copy of the set W^*.
(ii) For each edge uv of G let M_{uv} be a copy of M, and identify the copies of W^1 and W^2 in M_{uv} be identified vertexwise with W^u and W^v respectively.

Putting this all together. Now we show that $G \to T \iff G^* \to \mathcal{H}$. First, assume that $\phi : G^* \to \mathcal{H}$ is a homomorphism. Then one can check that

$$v \mapsto \phi|_{W^v} \in V(T)$$

is a homomorphism of $G \to T$.

On the other hand, assume that $\phi : G \to T$ is a homomorphism. Define $\phi' : G^* \to \mathcal{H}$ as follows. For each $v \in V(G)$ let ϕ' be equal to $\phi(g)$ on W^g, (recall that vertices of T are functions on W^*). For each edge uv of G, $\phi(u)\phi(v)$ is an edge of T, so there exists a homomorphism $\psi : \mathcal{H}^6 \to \mathcal{H}$ such that $\psi|_{W^u} = \phi(u)$ and $\psi|_{W^v} = \phi(v)$. So we can extend ϕ' to ψ on M_{uv}. Thus we extend ϕ' to a homomorphism $\phi' : G^* \to \mathcal{H}$.

□

References

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