Combinatorial Proof that Subprojective Constraint Satisfaction Problems are NP-Complete

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Abstract. We introduce a new general polynomial-time construction- the *fibre construction*- which reduces any constraint satisfaction problem CSP($\mathcal{H}$) to the constraint satisfaction problem CSP($\mathcal{P}$), where $\mathcal{P}$ is any subprojective relational structure. As a consequence we get a new proof (not using universal algebra) that CSP($\mathcal{P}$) is NP-complete for any subprojective (and thus also projective) relational structure. This provides a starting point for a new combinatorial approach to the NP-completeness part of the conjectured Dichotomy Classification of CSPs, which was previously obtained by algebraic methods. This approach is flexible enough to yield NP-completeness of coloring problems with large girth and bounded degree restrictions.

1 Introduction and Previous Work

Many combinatorial problems can be expressed as Constraint Satisfaction Problems (CSPs). This concept originated in the context of Artificial Intelligence (see e.g. [20]) and is very active in several areas of Computer Science. CSPs includes standard satisfiability problems and many combinatorial optimization problems, thus are also a very interesting class of problems from the theoretical point of view. The whole area was revitalized by Feder and Vardi [9], who reformulated CSPs as homomorphism problems (or H-coloring problems) for relational structures. Motivated by the results of [28] and [13], they formulated the following conjecture.

Conjecture 1. (Dichotomy) *Every Constraint Satisfaction Problem is either P or NP-complete.*

Schaefer [28] established the dichotomy for CSPs with binary domains, and Hell-Nešetřil [13] established the dichotomy for undirected graphs; it follows from [9] that the dichotomy for CSPs can be reduced to the dichotomy problem for H-coloring for oriented graphs. This setting, and related problems, have motivated

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intensive research in descriptive complexity theory. This is surveyed, for example, in [7], [13] and [11].

Recently the whole area was put in yet another context by Peter Jeavons and his collaborators, in [15] and [5], when they recast the complexity of CSPs into properties of algebras and polymorphisms of relational structures. Particularly, they related the complexity of CSPs to a Galois correspondence between polymorphisms and definable relations (obtained by Bodnarčuk et al. [1] and by Gaiger [10]; see [25] and [26]). This greatly simplified elaborate and tedious reductions of particular problems and led to the solution of the dichotomy problem for ternary CSPs [2] and other results which are surveyed, for example, in [5] and [12]. This approach to studying CSPs via certain algebraic objects yields, in particular, that for every *projective* structure H the corresponding CSP(H) is an NP-complete problem [16], [15]. The success of these general algebraic methods gave motivation for some older results to be restated in this new context. For example, [4] treats H-coloring problems for undirected graphs in such a way that the dichotomy between the tractable and NP-complete cases of H-coloring problem agrees with the general CSP Dichotomy Classification Conjecture stated in [6].

In this paper we propose a new approach to the dichotomy problem. We define a general construction- the *fibre construction*- which allows us to prove in a simple way that for every projective structure H, CSP(H) is NP-complete. In fact we define a *subprojective* structure and prove that for every subprojective relational structure H, CSP(H) is NP-complete. Though by an example of Ralph McKenzie [27], we know there are structures H that are not sub-projective for which CSP(H) is NP-complete, this is a first step in a combinatorial approach to the CSP Dichotomy Conjecture. In a later paper we extend this approach to include all structures that are known to be NP-complete, and possibly others. This will provide a new CSP Dichotomy Classification Conjecture, which is one of the main results yielded by algebraic methods. A discussion of this new Conjecture can be found in the full version of this paper, [23].

The fibre construction lends easily to restricted versions of CSPs, so allows us to address open problems from [8] and [17]. In particular, for any subprojective structure H, we show that CSP(H) is NP-complete for instances of bounded degree. Thus a fibre construction approach to a CSP Dichotomy Classification will reduce the Feder-Hell-Huang conjecture that NP-complete CSPs are NP-complete for instances of bounded degree to the CSP Dichotomy Classification Conjecture.

Our approach is motivated by the *Sparse Incomparability Lemma* [22] and *Müller's Extension Theorem* [21] (both these results are covered in [14]). These results are recalled and extended in Sect. 5. Strictly speaking, we do not need these results for our main results, Thm. 3 and Cor. 4, but they provided an inspiration for early forms of the fibre construction in [29] and [30] and for the general case presented here. Moreover, we do need these results to address the Dichotomy Conjecture for instances of large girth, extending results of [17].

The fibre construction is simple, and is a refinement of gadgets, or indicator constructions [13, 14], using familiar extremal combinatorial results [21, 22, 24]). However, the simplicity becomes obscured by the notation when dealing with general relational structures. Thus we find it useful to prove, in Sect. 3, only a simple case of the fibre construction. The case we prove is simple, but contains all the essential ingredients of the general fibre construction.

In Sect. 2 we introduce all the definitions and state the main results: Thm. 3 and Cor. 4. In Sect. 3 we prove a simple case of Thm. 3. In Sect. 4 we consider further applications of the fibre construction. Sect. 5 contains an extension of the some of the motivating results of our construction. Finally, in Sect. 6, we consider the relation of the fibre construction to the Dichotomy Classification Conjecture of [6].

2 Definitions and statement of results

We work with finite relational structures of a given type (or signature). A *type* is a vector $K = (k_i)_{i\in I}$ of positive integers, called *arities*. A *relational structure* $\mathcal{H}$ of type K, consists of a finite vertex set $V = V(\mathcal{H})$, and a k_i-ary relation $R_i = R_i(\mathcal{H}) \subset V^{k_1}$ on V, for each $i \in I$. An element of R_i is called an k_i-tuple. Thus a (di)graph is just a relational structure of type $K = (2)$. Its edges (arcs) are 2-tuples in the 2-ary relation R_1.

Given two relational structures $\mathcal{G}$ and $\mathcal{H}$ of the same type, an $\mathcal{H}$-*coloring of* $\mathcal{G}$ is a map $\phi : V(\mathcal{G}) \to V(\mathcal{H})$ such that for all $i \in I$ and every k_i-tuple $(v_1, \ldots, v_{k_i}) \in R_i(\mathcal{G})$, $(\phi(v_1), \ldots, \phi(v_{k_i}))$ is in $R_i(\mathcal{H})$. Fix a relational structure $\mathcal{H}$ (sometimes called *template*). CSP($\mathcal{H}$) is the following decision problem:

Problem CSP($\mathcal{H}$)
Instance: A relational structure $\mathcal{G}$;
Question: Does there exists an $\mathcal{H}$-coloring of $\mathcal{G}$?

We write $\mathcal{G} \to \mathcal{H}$ to mean that $\mathcal{G}$ has an $\mathcal{H}$-coloring.

A relational structure $\mathcal{H}$ is a *core* if its only $\mathcal{H}$-colorings are automorphisms.

It is well known, (see, for example, [14]) that $\mathcal{G} \to \mathcal{H}$ if and only if $\mathcal{G}' \to \mathcal{H}'$, where $\mathcal{G}'$ and $\mathcal{H}'$ are the cores of $\mathcal{G}$ and $\mathcal{H}$ respectively. Therefore, in the sequel, we only consider relational structures that are cores.

All relational structures of a given type form a category with nice properties. In particular, this category has products and powers which are defined explicitly as follows:

Given a relational structure $\mathcal{H}$, and a positive integer d, the d-ary *power* $\mathcal{H}^d$ of $\mathcal{H}$ is the relational structure of the same type as $\mathcal{H}$, defined as follows.

- $V(\mathcal{H}^d) = \{(v_1, \ldots, v_d) \mid v_1, \ldots, v_d \in V(\mathcal{H})\}$.
- For $i \in I$, $((v_{1,1}, v_{1,2}, \ldots, v_{1,d}), \ldots, (v_{k_i,1}, \ldots, v_{k_i,d}))$ is in $R_i(\mathcal{H}^d)$ if and only if all of $(v_{1,1}, v_{2,1}, \ldots, v_{k_i,1}), \ldots, (v_{1,d}, \ldots, v_{k_i,d})$ are in $R_i(\mathcal{H})$.

An $\mathcal{H}$-coloring of $\mathcal{H}^d$ (i.e. a homomorphism $\mathcal{H}^d \to \mathcal{H}$) is called a *d-ary polymorphism* of $\mathcal{H}$. A d-ary polymorphism ϕ is called a *projection* if there exists some $i \in 1, \ldots, d$ such that $\phi((v_1, \ldots, v_d)) = v_i$ for any $v_1, \ldots, v_d \in V(\mathcal{H})$. Let $\mathrm{Pol}(\mathcal{H})$, $\mathrm{Aut}(\mathcal{H})$ and $\mathrm{Proj}(\mathcal{H})$ be the sets of polymorphisms, automorphisms and projections (of all arities) of $\mathcal{H}$. A relational structure $\mathcal{H}$ is *projective* if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, $\phi = \sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. (It is shown in [19] that almost all relational structures are projective.)

The following definition of graphs that are, in a sense, locally projective, is our principal definition.

Definition 2. *A subset S of $V(\mathcal{H})$ is called* projective *if for every $\phi \in \mathrm{Pol}(\mathcal{H})$, ϕ restricts on S to the same function as does $\sigma \circ \pi$ for some $\sigma \in \mathrm{Aut}(\mathcal{H})$ and some $\pi \in \mathrm{Proj}(\mathcal{H})$. S is called* non-trivial *if $|S| > 1$. A relational structure $\mathcal{H}$ is called* subprojective *if it is a core and it contains a non-trivial projective subset.*

Note that any subset of a projective set is again projective. A structure is projective if and only if the set of all its vertices is projective. It is easy to find subprojective structures which fail to be projective.

The main tool of the paper is the following general indicator construction which we call the *fibre construction*. This construction extends a construction first used in a in a Ramsey theory setting in [29], and then proved in [30] in the present form, for $\mathcal{H} = K_3$ and $\mathcal{P}$ being projective. A special case of it is proved in Sect. 3, the full proof is relegated to the full version of the paper.

Theorem 3. *Let $\mathcal{H}$ be any relational structure, and let $\mathcal{P}$ be any subprojective relational structure. Then there exists a polynomial time construction, the fibre construction, $\mathcal{M}^{\mathcal{P}}_{\mathcal{H}}$ which provides for any instance $\mathcal{G}$ of $\mathrm{CSP}(\mathcal{H})$, an instance $\mathcal{M}^{\mathcal{P}}_{\mathcal{H}}(\mathcal{G})$ of $\mathcal{P}$ such that*

$$\mathcal{G} \to \mathcal{H} \iff \mathcal{M}^{\mathcal{P}}_{\mathcal{H}}(\mathcal{G}) \to \mathcal{P}.$$

Note that $\mathcal{H}$ and $\mathcal{P}$ need not be of the same type. Since $\mathrm{CSP}(K_3)$ is NP-complete, taking $\mathcal{H}$ in to be K_3 gives the following result.

Corollary 4. *For any subprojective relational structure $\mathcal{P}$, the problem $\mathrm{CSP}(\mathcal{P})$ is NP-complete.*

The fibre construction also has immediate applications to restricted versions of CSP complexity.

The *degree* of a vertex v in a relational structure $\mathcal{G}$ is the number of tuples it occurs in in $\bigcup R_i$, and the maximum degree, over all vertices in $\mathcal{H}$, is denoted by $\Delta(\mathcal{G})$. $\mathcal{G}$ is called *b-bounded* if $\Delta(\mathcal{G}) \leq b$.

It is conjectured in [8] that for any relational structure $\mathcal{H}$, if $\mathrm{CSP}(\mathcal{H})$ is NP-complete, then there is some finite b such that $\mathrm{CSP}(\mathcal{H})$ is NP-complete when restricted to b-bounded instances.

In [30], this was shown to be true in the case of graphs and projective relational structures $\mathcal{H}$. Further explicit bounds were given on $b(\mathcal{H})$, which is the minimum b such that $\mathrm{CSP}(\mathcal{H})$ is NP-complete when restriced to b-bounded instances. In Sect. 4, we observe the following corollary of the proof of Thm. 3.

Corollary 5. *For any subprojective relational structure $\mathcal{P}$,*

$$b(\mathcal{P}) < (4 \cdot \Delta(\mathcal{P})^6).$$

This greatly improves the bound on $b(H)$ from [30] in the case of subprojective graphs H. We intend, in a later paper to show that all non-bipartite graphs are subprojective, thus applying this better bound to all graphs.

Degrees and short cycles are classical restrictions for coloring problems. Recall that *girth* $g(G)$ of a graph G is the length of the shortest cycle in G. We then observe that the following result about sparse graphs follows from our extension (from Sect. 5) of the *Sparse Incomparability Lemma* [22], [24].

Theorem 6. *Let H be a subprojective graph, and ℓ a positive integer. Then the problem* CSP(H) *is NP-complete when restricted to graphs with girth $\geq \ell$.*

This addresses a problem of [17] where the question of CSPs when restricted to instances with large girth was studied. This result can be generalized futher to relational structures but we decided to stop here.

3 The Fibre Construction

So called indicator constructions are often used relate the conplexity of different CSPs. The basic idea is that one can reduce CSP(H) to CSP(H') by constructing in polynomial time, for any instance G of CSP(H), an instance G' of CSP(H'), such that

$$G \to H \iff G' \to H'.$$

If CSP(H) is NP-complete, then CSP(H') must also be NP-complete. See the proof of the H-coloring dichotomy in [13] for an intricate use of such constructions.

One of the difficulties with indicator constructions is that one uses many ad hoc tricks to find a constrution for a particular graphs H' or H. The fibre construction, Thm. 3, is an indicator construction that will suffice for all reductions.

In this section, we prove the following simple case of the fibre construction.

Proposition 7. *There exists a polynomial time construction which provides for and graph G, a graph $M(G)$ such that*

$$G \to C_5 \iff M(G) \to K_3.$$

Ours is not the most elegant known reduction of C_5-coloring to K_3-coloring, but it has the advantage that it can be easily generalized. After the proof, we discuss a couple of issues that we must deal with in the general case, Thm. 3. The proof of the general case can be found in the full version of the paper.

3.1 Notation

Given an indexed set $W^* = [w_1^*, \ldots, w_d^*]$ of vertices, *a copy* W^a *of the set* W^* will mean the indexed set $W^a = [w_1^a, \ldots, w_d^a]$. Given two copies W^a and W^b of the same set W^* we say that we *identify* W^a *and* W^b *index-wise* to mean we identify the vertices w_i^a and w_i^b for $i = 1, \ldots, d$. When we define a function f on W^*, we will assume it to be defined on any copy W^a of W^* by $f(w_\alpha^a) = f(w_\alpha^*)$ for all $\alpha = 1, \ldots, d$. We will often refer to a function f on an indexed set W^* as a *pattern* of W^*. In the case that the image of f is contained in the vertex set of some graph H we speak about *H-pattern* of W^*.

3.2 The Fibre Gadget

The construction consists of two parts. In the first part we build a fibre gadget M which depends only on C_5 and K_3. To build the fibre gadget M we need the following simple lemma which is motivated by a result of Müller, [21].

Lemma 1. *Let $\mathcal{P}$ be a subprojective relational structure with non-trivial projective subset S. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be a set of S patterns of W satisfying the following condition (*).*

> *For any pair $w \neq w' \in W$, there exists some $\gamma \in \Gamma$ for which $\gamma(w) \neq \gamma(w')$.*

Then there exists a relational structure $\mathcal{M}$, isomorphic to $\mathcal{P}^d$, with $W \subset V(\mathcal{M})$, such that the set of $\mathcal{P}$-colorings of $\mathcal{M}$, when restricted to W, is exactly

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(\mathcal{P}), \gamma \in \Gamma\}.$$

Proof (Proof of Lemma 1). Put $\mathcal{M} = \mathcal{P}^d$ and for each $w \in W$, identify w with the vertex $(\gamma_1(w), \ldots, \gamma_d(w))$ of $\mathcal{M}$. By condition (*), these are distinct elements of $V(\mathcal{M})$.

Since S is a projective subset of $\mathcal{P}$, the only $\mathcal{P}$-colorings of $\mathcal{M} = \mathcal{P}^d$ restrict on S^d, which contains W, to $\alpha \circ \pi$ where α is a automorphism of $\mathcal{P}$ and π is a projection. But the projections restrict on W to exactly the maps of Γ, so the lemma follows.

The following lemma provides the fibre gadget M.

Lemma 2. *There exists a graph M containing two copies W^a and W^b of an indexed set W^*, and a set $F = \{f_x \mid x \in V(C_5)\}$ of distinct K_3-patterns of W^*, such that the following conditions are true upto some permutation of $V(K_3)$.*

i. Any K_3-coloring of M, restricted to W^a, (or to W^b) is in F.
ii. For any K_3-coloring ϕ of M, ϕ restricts on W^a to f_x and on W^b to f_y for some edge xy of C_5.
iii. For any edge xy (or yx) of C_5, there is a K_3-coloring ϕ of M that restricts on W^a to f_x and on W^b to f_y.

Moveover, $M \cong (K_3)^{10}$.

Proof. Let $V(K_3) = \{0, 1, 2\}$. Let $W^* = [w^*_x \mid x \in V(C_5)]$, and let $F = \{f_x \mid x \in V(C_5)]$, where f_x is the $\{0, 1\}$-pattern (K_3-pattern) defined by

$$f_x(w^*_y) = \begin{cases} 1 & \text{x = y} \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = W^a \cup W^b$, where W^a and W^b are disjoint copies of W^* and let $\Gamma = \{\gamma_{xy} \mid xy \in E(C_5)\}$ where γ_{xy} is the $\{0, 1\}$-pattern of W defined by

γ_{xy} restricted to W^a is f_x and restricted to W^b is f_y.

Observe that γ_{xy} and γ_{yx} are distinct elements of Γ for every edge xy of C_5, so $|\Gamma| = 10$.

Apply Lem. 1 to W and Γ. The instance $\mathcal{M}$ of CSP(K_3) that it returns is clearly the graph M that we are looking for.

The name 'fibre gadget' comes from the relation of the vertices of W^* to the set of K_3-patterns F. We view $w \in W^*$ as a fibre in $V(K_3)^{|F|}$, whose i^{th} postition corresponds to its image under the ith pattern f_{x_i} of F.

3.3 The Fibre Construction

The fibre gadgets are put together with the following construction, which call the fibre construction.

Construction 8. *Let W^*, F, and M be as in Lem. 2. Let G be an instance of* CSP(K_3)*, and construct $M(G)$ as follows. (See Fig. 1.)*

i. For each vertex v of G let W^v be a copy of W^*.
ii. For each edge uv of G let M_{uv} be a copy of M. Index-wise, identify W^u and W^v with the copies of W^a and W^b, respectively, in M_{uv}.

Thus $M(G)$ consists of $|V(G)|$ copies of W^* and $|E(G)|$ copies of M. All vertices are distinct unless identified above.

We can now prove Prop. 7.

Proof. For any graph G let $M(G)$ be the graph defined by the fibre construction, Const. 8. As $M(G)$ is made of $|E(G)|$ copies of M, which is independent of G, this is a polynomial time construction.

Let ϕ be a K_3-coloring of $M(G)$. We show that this defines a C_5 colouring ϕ' of G. It is enough to show this for a compontent of $M(G)$. Now ϕ restricts on W^v, for each vertex v of G, to $\sigma \circ f$ for some permutation σ of $V(K_3)$ and some pattern f in F. Since the number of vertices of each color is constant over all patterns of F, and this property is not preserved under any permutaion of $V(K_3)$, the permutaion σ must be constant for all W^v. We assume that it is the identity permutation, so ϕ restricts on each W^v to some pattern f in F. Thus

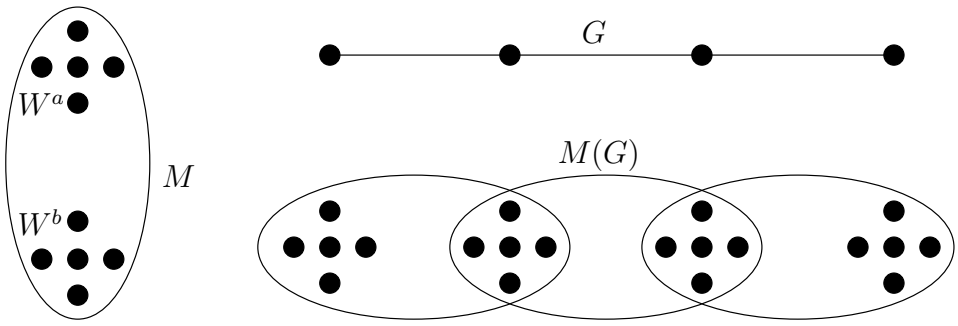

Fig. 1. Fibre Construction

$\phi' : V(G) \to V(C_5)$ is well defined by letting $\phi'(v) = x$ where ϕ restricts on W^v to the pattern f_x. Moreover, by property (ii) of Lem. 2, ϕ' is a C_5-coloring of G.

On the other hand, given a C_5-coloring ϕ' of G we define a K_3-coloring ϕ of $M(G)$ as follows. For all vertices v of G, let ϕ be $f_{\phi'(v)}$ on the set W^v. For every edge uv of G, the sets W^u and W^v are already colored by ϕ, and we must extend this coloring to M^{uv}. Now ϕ restricts on W^u to $f_{\phi'(u)}$ and on W^v to $f_{\phi'(v)}$, and $\phi'(u)\phi'(v)$ is an edge of C_5, so by property (iii) of Lem. 2 ϕ can be extended to M^{uv}. Thus ϕ can be extended to a K_3-coloring of $M(G)$.

3.4 Remark

This outline gives only the idea of the general proof. There are several obstacles. For example, in the general case of relational structures, we will need a different fibre gadget for each kind of relation. And, of course, our relations need not be symmetric. In the general case, the set F from Lem. 2 will be S-patterns, instead of K_3-patterns. Also, for general $\mathcal{H}$, it may be more difficult to to ensure that Γ in the proof of Lem. 2 satisfies property (*) of Lem. 1. We thus use a more general version on Lem. 1 in which Γ need not satisfy (*), but which returns a $\mathcal{M}$ that is not necessarily isomorphic to $\mathcal{P}^d$.

These are just technicalities which can be handled with care.

4 Applications

4.1 Degree Bounded CSPs

We mentioned in the introduction, that because CSP(K_3) is NP-complete, taking $\mathcal{H} = K_3$, Cor. 4 follows from Thm. 3. In fact, CSP(K_3) is NP-complete for 4-bounded instances G.

For a 4-bounded graph G, the fibre construction would yield an instance $\mathcal{M}^{\mathcal{P}}_{K_3}(G)$ of CSP($\mathcal{P}$) with maximum degree $(4 \cdot \Delta(\mathcal{P}))^6$. Thus Cor. 5, follows from the proof of Thm. 3.

4.2 Girth Restricted H-coloring

The results in this subsection are for graphs.

The following lemma is proved in [21] in the case that P is a complete graph, and is proved in [24] without item (iii) in the case that P is projective. In both of these cases, $S = V(P)$.

Lemma 3. *Let P be a subprojective graph with projective subset S, and let $\ell \geq 3$ be an integer. Let W be an indexed set, and let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ be a set of S patterns of W. Then there exists a relational structure M with $W \subset V(M)$, such that the following are true:*

i. The set of P-colorings of M, when restricted to W, is exactly

$$\{\alpha \circ \gamma \mid \alpha \in \mathrm{Aut}(P), \gamma \in \Gamma\}.$$

ii. M has girth at least ℓ.
iii. The distance, in M, between any two vertices of W is at least g.

Proof. This lemma follows from Thm. 11 which is a local form of the main result of [24]. The result will be stated in Sect. 5.

Using this in place of Lem. 1 in the fibre construction, we can ensure that the graph $\mathcal{M}_H^P(G)$ that is returned has girth ℓ. Thus Thm. 6 follows.

4.3 Conservative CSPs

A constraint satisfaction problem CSP($\mathcal{H}$) is *conservative* if $\mathcal{H}$ contains all possible unary relations. Such a CSP is also known as List $\mathcal{H}$-colouring.

In [3], Bulatov proves the following dichotomy for conservative CSPs.

Theorem 9. *[3] A conservative constraint satisfaction problem* CSP($\mathcal{H}$) *is NP-complete if there there is a set $B \subset V(\mathcal{H})$ of size at least 2 such that for any polymophism ϕ of $\mathcal{H}$, ϕ restricted to B is essentially unary. Otherwise,* CSP($\mathcal{H}$) *is polynomial time solvable.*

The difficult part of Bulatov's paper is the polynomial time solvable part of this result. The NP-complete part follows quickly from the algebraic approach of [15] and [5]. We observe that the NP-complete part is also immedieate from Cor. 4. Indeed, since we only consider cores $\mathcal{H}$, any essentially unary operation on B is $\alpha \circ \pi$ where π is a projection of B^d to B and α is an automorphism of $\mathcal{H}$. Thus B is a projective subset of $\mathcal{H}$.

5 Coloring Theorems - Combinatorial Background

The main motivation for our construction is a result of Müller [21], which is a special graph case of Lem. 1, except that it returns a graph M of arbitrary girth. The difficult part of the lemma is, of course, ensuring that M has arbitrary girth.

He did this with a special case of Lem. 11. Müller's lemma was extended in [24], and the form here, is a localisation of their version.

Localising the notion of H-pointed graphs H, from [24], we get the following definition.

Definition 10. *Let H, H' be graphs. Subsets S of $V(H)$ and S' of $V(H')$ are said to be* (H, H')-pointed subsets *if for any two homomorphisms $g, g' : H \to H'$ which satisfy $g(x) = g'(x) \in S'$, whenever $x \neq x_0$ and $x \in S$ (for some fixed vertex $x_0 \in S$), then $g(x_0) = g'(x_0) \in S'$.*

Theorem 11. *For every graph H and every choice of positive integers k and l there exists a graph G together with a surjective homomorphism $c : G \to H$ with the following properties.*

i. $g(G) > l$;
ii. *For every graph H' with at most k vertices and there exists a homomorphism $g : G \to H'$ if and only if there exists a homomorphism $f : H \to H'$.*
iii. *For every (H, H')-pointed subsets $S \subset V(H), S' \subset V(H')$ with at most k vertices and for every homomorphism $g : G \to H'$ holds: if homomorphisms $f, f' : H \to H'$ satisfy $g = f \circ c$, then $f(x) = f'(x)$ for every $x \in S$.*

The proof of Thm. 11 is along the same lines as less general version proved in [24]. We essentially repeat their proof replacing the notion of H-pointed with its localization, (H, H')-pointed. This is routine but lengthy, so we omit the proof.

6 CSP Dichotomy Classification Conjecture

In [5], the universal algebra approach of [15] is extended to to show that CSP($\mathcal{H}$) is NP-complete for a large class of CSPs. A conjecture is made that CSP($\mathcal{H}$) is polynomial time solvable for all other CSPs $\mathcal{H}$. In [18], this conjecture is then transported to the language of posets.

An algebra $\mathcal{A} = (A, F)$ consists of a non-empty set A, and a set F of finitary operations on A. It is *finite* if A is finite. Given a relational structure $\mathcal{H}$, recall that Pol($\mathcal{H}$) is the set of polymorphisms of $\mathcal{H}$. This defines an algebra $\mathcal{A}_{\mathcal{H}} = (V(\mathcal{H}), \mathrm{Pol}(\mathcal{H}))$. We say that $\mathcal{A}_{\mathcal{H}}$ is NP-complete if CSP($\mathcal{H}$) is.

The following two definitions are borrowed directly from [5].

Definition 12. *Let $\mathcal{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f \in F$ and for any $b_1, \ldots, b_d \in B$, where d is the arity of f, we have $f(b_1, \ldots, b_d) \in B$. Then the algebra $\mathcal{B} = (B, F|_B)$ is called a subalgebra of $\mathcal{A}$, where $F|_B$ consists of the restrictions of all operations in F to B.*

Definition 13. *Let $\mathcal{B} = (B, F_1)$ and $\mathcal{C} = (C, F_2)$ be such that $F_1 = \{f_i^1 \mid i \in I\}$ and $F_2 = \{f_i^2 \mid i \in I\}$, where both f_i^1 and f_i^2 are d_i-ary, for all $i \in I$. Then $\mathcal{C}$ is a homomorphic image of $\mathcal{B}$ if there exists a surjection $\psi : B \to C$ such that the following identity holds for all $i \in I$, and all $b_1, \ldots, b_{d_i} \in B$.*

$$\psi \circ f_i^1(b_1, \ldots, b_{d_i}) = f_i^2(\psi(b_1), \ldots, \psi(b_{d_i})).$$

Given an algebra $\mathcal{C} = (C, F)$, the *term operators* of $\mathcal{C}$ refer to the set of finitary operators of C that preserve the same relations on C as F does. Thus all operators in F are term operators. A d-ary operator f of F is *essentially unary* if $f = f' \circ \pi$ for some projection $\pi : C^d \to C$ and some non-constant function $f' : C \to C$. Because this f' is non-constant, if F has any essentially unary operators, then $|C| \geq 2$.

The following result is Cor. 7.3 in [5].

Theorem 14. *A finite algebra $\mathcal{A}$ is NP-complete if it has a subalgebra $\mathcal{B}$ with a homomorphic image $\mathcal{C}$, all of whose term operators are essentially unary.*

Further, they conjecture that CSP($\mathcal{H}$) is NP-complete for a relational structure $\mathcal{H}$, only if it is NP-complete by the above theorem.

We at first thought that any relational structure $\mathcal{H}$ such that $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, all of whose term operators are essenially unary, was subprojective. However, Ralph McKenzie [27] provided us with an explicit counterexample to this fact. Further, he showed that for any subprojective structure $\mathcal{H}$ that $\mathcal{A}_{\mathcal{H}}$ has a subalgebra which has a homomorphic image, all of whose term operators are essentially unary.

This shows that subprojective structures are certainly not the only structures yielding NP-complete CSPs. However, the fibre construction can be adapted to show NP-completeness for much more than subprojective structures. In a future paper we extend the fibre construction to show, at least, that all $\mathcal{H}$ such that $\mathcal{A}_{\mathcal{H}}$ has a subalgebra $\mathcal{B} = \mathcal{A}(B, F_1 = \mathrm{Pol}(\mathcal{H})|_B)$ with a homomorphic image $\mathcal{C} = \mathcal{A}(C, F_2)$, are NP-complete. A discussion of this extension appears in [23], the full version of this paper, and a full presentation will appear in a future paper.

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