NU POLYMORPHISMS ON REFLEXIVE DIGRAPHS

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Abstract. We find a set of generators of the variety of reflexive digraphs admitting k – NU polymorphisms. We do this, in spite of the fact that such digraphs do not have finite tree duality, by defining finite duals of infinite trees. As a result of this, we answer a question of Quackenbush, Rival, and Rosenberg, giving a finite family of generators of the variety of finite bounded posets admitting k – NU polymorphisms.

1. Introduction

The present paper continues the papers [7] and [8]. In [7] the class of reflexive graphs admitting compatible k – NU operations was described in terms of a set of generators from which it is built via finite products and retractions. In [8] the same was done for irreflexive graphs. We refer the reader to [7] for a thorough discussion of the motivations for studying such operations on graphs.

In the present paper, we describe generators of the set of reflexive digraphs admitting compatible k – NU operations. This is a generalisation of the results of [7], and the construction of the generators follows the general attack of that paper, taking the duals of vertex-coloured trees and considering the substructures induced by their reflexive vertices. However, the present paper differs in one very important aspect. Whereas for a reflexive (or irreflexive) graph $\mathbb{H}$, the existence of an NU polymorphism is equivalent to the existence of a finite tree duality, the same is not true for reflexive digraphs.

To get a set of generators from which we can describe the class of reflexive NU digraphs via finite products and retractions, we define an *infinite tree*, a type of limit object for a family of trees, and construct a (finite) dual of an infinite tree by taking a limit of the duals of trees in the family.

In Section 2 we give basic definitions and notation necessary to define and use the known tools we introduce in Section 3– critical obstructions and dualities. In Section 4 we define a reflexive duality– a fairly transparent augmentation of the dualities provided by results of Section 3, which serves mostly to simplify presentation. In Section 5 we give an outline of the proof of our main result; the reader somewhat familiar with the area could easily jump forward to this section now. Section 6 is where we do most of our work, we define infinite trees and their duals, and then prove that they are, in fact, the duals that we claim they are.

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In Section 7 we show that infinite dualities of trees can be replaced with finite dualities of infinite trees. Using this and the duals constructed in Section 6, we get, in Section 8, a finite presentation of a reflexive $k-$NU digraph: we show that a reflexive digraph is NU if and only if it is retract of a finite product of finite generators. We then observe that for graphs of bounded diameter, the set of generators we must consider is also finite.

As a by-product of our proof, we observe in Section 9 that a reflexive digraph with constants has finite duality if and only if it is strongly connected and admits an NU operation.

In Section 10 we restrict our results to the case of acyclic transitive reflexive digraphs, or posets. We see that our generators are themselves posets, and so with not much work, we answer a question of [20], and show that there is a finite family of posets such that every finite, bounded, $k-$NU poset is a retract of a finite product of posets from the family. As it requires some technical definitions that we do not use elsewhere, we defer the background discussion of this application until Section 10.

2. Preliminaries and Notation

A digraph $\mathbb{H}$ is reflexive if all vertices have loops. Though our main results are about reflexive digraphs, the tools we use require us to consider the more general structure of coloured digraphs. We define these in a slightly round-about way through the language of relational structures, as this is the language these tools are given in.

We refer the reader to [1] for basic notation and terminology. In the present paper we will use blackboard fonts such as $\mathbb{G}$, $\mathbb{H}$, etc. to denote relational structures and their latin equivalent G, H, etc. to denote their respective *universes*, or vertex sets. A *signature* τ is a (finite) set of relation symbols with associated arities. We say that $\mathbb{H} = \langle H; R(\mathbb{H})(R \in \tau)\rangle$ is a *relational structure of signature* τ if $R(\mathbb{H})$ is a relation on H of the corresponding arity, for each relation symbol $R \in \tau$.

Let $\mathbb{G}$ and $\mathbb{H}$ be structures of signature τ. A *homomorphism* from $\mathbb{G}$ to $\mathbb{H}$ is a map f from G to H such that, for all $R \in \tau$, if $(g_1, \ldots, g_m) \in R(\mathbb{G})$ then $(f(g_1), \ldots, f(g_m)) \in R(\mathbb{H})$. We write $\mathbb{G} \to \mathbb{H}$ to indicate there exists a homomorphism from $\mathbb{G}$ to $\mathbb{H}$. A homomorphism $r : \mathbb{H} \to \mathbb{R}$ is a *retraction* if there exists a homomorphism (called a *coretraction*) $e : \mathbb{R} \to \mathbb{H}$ such that $r \circ e$ is the identity on $\mathbb{R}$ and we say that $\mathbb{R}$ is a *retract* of $\mathbb{H}$ and write $\mathbb{R} \trianglelefteq \mathbb{H}$. A structure $\mathbb{H}$ is called a *core* if every homomorphism from $\mathbb{H}$ to itself is a permutation of H; note that a retract of $\mathbb{H}$ with universe of minimum cardinality is a core, and is unique up to isomorphism, and hence we may speak of *the core* of the structure $\mathbb{H}$. Throughout this paper we consider the usual product $\prod_{i\in I} \mathbb{H}_i$ of τ-structures $\mathbb{H}_i$. That is, the universe of $\prod_{i\in I} \mathbb{H}_i$ is the cartesian product $\prod_{i\in I} H_i$; we denote elements as functions $x : I \to \bigcup_{i\in I} H_i$ with $x(i) \in H_i$. For each R in τ, $(x_1, \ldots, x_m) \in R(\prod_{i\in I} \mathbb{H}_i)$ if and only if $(x_1(i), \ldots, x_m(i)) \in R(\mathbb{H}_i)$ for all $i \in I$. We shall consider notations such as $\prod_{i=1}^{n} \mathbb{H}_i$ and $\mathbb{H}^k$ to be self-evident.

Let H be a non-empty set, let θ be an m-ary relation on H and let $f : H^k \to H$ be a k-ary operation on H. We say that f *preserves* θ if the following holds: if the $m \times k$ matrix M has each column in θ, then applying f to the rows of M yields a tuple of θ. If $\mathbb{H}$ is a τ-structure, and f preserves each of its basic relations (equivalently, if f is a homomorphism from $\mathbb{H}^k$ to $\mathbb{H}$), we say that f is *compatible*

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with $\mathbb{H}$, or that $\mathbb{H}$ *admits* f; one also says that f is a *polymorphism* of $\mathbb{H}$, see [1] and [19] for instance. Recall that for $k \geq 3$, f is a k-ary near-unanimity ($k-\mathrm{NU}$) operation if it satisfies, for every $1 \leq i \leq k$, the identity

$$f(x, \ldots, x, \underbrace{y}_{i}, x, \ldots, x) = x.$$

A structure is said to be $k-\mathrm{NU}$ if it admits a $k-\mathrm{NU}$ polymorphism.

The following is standard for finite products, and the proof for infinite products is just as easy.

Lemma 2.1. *Let $\mathbb{H}_i$ for $i \in I$ be $k-\mathrm{NU}$ structures, then $\prod_I \mathbb{H}_i$ is also $k-\mathrm{NU}$.*

Proof. Let f_i be a $k-\mathrm{NU}$ polymorphism on f_i for each $i \in I$, and let $f = \prod_{i \in I} f_i$ be the product map; that is let it be defined for $x_1, x_2 \ldots, x_k \in \prod_I H_i$ by

$$f(x_1, \ldots, x_k)(i) = f_i(x_1(i), x_2(i), \ldots, x_k(i)).$$

It is trivial to check that this is NU. □

As the composition of an NU-operation with a retraction is still NU, we have

Lemma 2.2. *If $\mathbb{H}$ is $k-\mathrm{NU}$, then any retract of $\mathbb{H}$ is also $k-\mathrm{NU}$.*

Another standard proof, see for example [2], gives the following, which allows us to restrict our view to connected structures.

Lemma 2.3. *A structure $\mathbb{H}$ is $k-\mathrm{NU}$ if and only if every connected component of $\mathbb{H}$ is.*

2.1. **H- digraphs.** Our main signature is that of a digraph, consisting of one binary relation A whose elements we refer to as *arcs*. But as mentioned above, we will have to expand the signature a bit.

A *digraph with constants* is a relational structure $\mathbb{H}$ with a binary relation A, and for each vertex $h \in H$ a unary relation $S_h = \{h\}$, which is called a *constant* relation.

Any structure $\mathbb{G}$ of the same signature $\tau_{\mathbb{H}} = \{A\} \cup \{S_h \mid h \in H\}$ is called an H-digraph. Note that the structure of an H-digraph depends only on the set H, the structure of $\mathbb{H}$ is irrelevant. The vertices of $\mathbb{G}$ in $S_h(\mathbb{G})$ are said to be *coloured with*, or to have, the *colour* h. If the arc relation $A(\mathbb{G})$ of an H-digraph contains neither loops nor cycles, directed or otherwise, then $\mathbb{G}$ is called an *H-tree*.

If $\mathbb{H}$ is a digraph, then $\mathbb{H}^{+c}$ is the digraph with constants we get by adding all constant relations. If $\mathbb{H}$ is a digraph with constants, then $\mathbb{H}^{-c}$ is the digraph we get by removing constant relations.

As for an operation $f : \mathbb{H}^d \to \mathbb{H}$, preserving constants simply means that the operation is idempotent, and since NU operations are by definition idempotent, we have the following.

Fact 2.4. *A digraph $\mathbb{H}$ with constants is $k-\mathrm{NU}$ if and only if $\mathbb{H}^{-c}$ is $k-\mathrm{NU}$.*

Observe that for an H-digraph $\mathbb{G}$, a homomorphism $\mathbb{G} \to \mathbb{H}$ is a digraph homomorphism that takes any h-coloured vertex of $\mathbb{G}$ to the vertex h of $\mathbb{H}$. As such, the problem of deciding if $\mathbb{G}$ admits a homomorphism to $\mathbb{H}$ is the problem known as $\mathbb{H}$-*precolour extension*, and is easily reduced to and from the problem of *retraction* to $\mathbb{H}$; for more detail see [6]. A homomorphism $\mathbb{H} \to \mathbb{G}$ is a digraph homomorphism that takes any vertex h of $\mathbb{H}$ to one of the h-coloured vertices of $\mathbb{G}$, so the

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problem of deciding if $\mathbb{H}$ admits a homomorphism to $\mathbb{G}$ is an instance of the *list $\mathbb{G}$-homomorphism problem*.

3. NU Operations and Duality

All of the main results and definitions in this section are known in more generality. We have specialised them to our purposes.

Definition 3.1. A *duality* for a digraph $\mathbb{H}$ with constants is a family $\mathcal{D}$ of H-digraphs such that for any H-digraph $\mathbb{G}$, the following holds.

$$(\forall \mathbb{D} \in \mathcal{D}, \mathbb{D} \not\to \mathbb{G}) \iff \mathbb{G} \to \mathbb{H}.$$

We list some facts that are immediate from the definition by simple arrowing arguments.

Lemma 3.2. *Let $\mathcal{D}_i$ be a duality for $\mathbb{H}_i$, for every $i \in I$. The following all hold.*

(i) $\bigcup_{i\in I} \mathcal{D}_i$ *is a duality for* $\prod_{i\in I} \mathbb{H}_i$
(ii) If $\mathbb{H} \trianglelefteq \prod_{i\in I} \mathbb{H}_i$, *then* $\mathcal{D} = \bigcup_{i\in I} \mathcal{D}_i$ *is a duality for* $\mathbb{H}$.
(iii) If $\mathcal{D}'$ *is a family such that for every* $\mathbb{T}$ *in* $\mathcal{D}$ *there is some* $\mathbb{T}' \in \mathcal{D}'$ *with* $\mathbb{T}' \to \mathbb{T}$, *and such that no* $\mathbb{T}'$ *in* $\mathcal{D}'$ *maps to* $\mathbb{H}$, *then* $\mathcal{D}'$ *is also a duality for* $\mathbb{H}$.

Proof. (i) By the fact that projections are homomorphisms and using the standard property of direct products, we have that $\mathbb{T} \to \prod_{i\in I} \mathbb{H}_i$ if and only if $\mathbb{T} \to \mathbb{H}_i$ for all $i \in I$, which holds precisely when no member of the union of the dualities maps to $\mathbb{T}$.

The statement (ii) follows from (i) by the fact that a structure is homomorphically equivalent to any retract of it. Statement (iii) is trivial. □

Definition 3.3. A *critical obstruction* of a digraph $\mathbb{H}$ with constants is an H-digraph $\mathbb{T}$ such that $\mathbb{T} \not\to \mathbb{H}$, but $\mathbb{T}' \to \mathbb{H}$ for any proper substructure $\mathbb{T}'$ of $\mathbb{T}$.

The following is clear.

Fact 3.4. *The family of all critical obstructions for a digraph $\mathbb{H}$ with constants is a duality for $\mathbb{H}$.*

It turns out that the existence of an NU operation on $\mathbb{H}$ is related to the number of colours in the critical obstructions. The following adaptation of a result of [23] was proved in [7].

Theorem 3.5. *A digraph $\mathbb{H}$ with constants is $k-$NU if and only if none of its critical obstructions has more than $k-1$ coloured vertices.*

Further, if $\mathbb{H}$ is reflexive and has an NU operation, then it has a duality of critical obstructions each of which is a tree. Indeed, this comes from Lemma 3.2(iii) by the following result of [14]. (It is proved in the proof of Theorem 3.10 of [14]. The statement there is for reflexive digraphs G, but their 'G-obstruction', which is our 'critical obstruction', is a G-digraph, so it corresponds to a statement about digraphs with constants.)

Theorem 3.6 ([14])**.** *Let $\mathbb{H}$ be a reflexive digraph with constants. If $\mathbb{H}$ admits an* NU *operation, then for any critical obstruction $\mathbb{D}$ of $\mathbb{H}$ there is another critical obstruction $\mathbb{T}$ of $\mathbb{H}$ such that $\mathbb{T}$ is an H-tree, and $\mathbb{T} \to \mathbb{D}$.*

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From Fact 3.4 and Theorems 3.5 and 3.6 the following is immediate.

Corollary 3.7. *A reflexive digraph with constants $\mathbb{H}$ is $k-$NU if and only if it has a duality consisting of critical H-trees, each having at most $(k-1)$ coloured vertices.*

Finally, H-trees in this duality have well behaved colours.

Definition 3.8. An H-tree is *elementary* if it is a core and it has one colour on each leaf and no other colours, or if it is a single vertex with two colours.

The following lemma should be clear. With the trivial observation that a critical obstruction must be a core, the proof is exactly the same as the easy proof in the case that $\mathbb{H}$ is a reflexive symmetric graph, which can be found in the proof of Theorem 3.9 of [7].

Lemma 3.9. *Any H-tree that is a critical obstruction for a reflexive digraph $\mathbb{H}$ with constants, is an elementary H-tree.*

Our other main tool is the following result of Nešetřil and Tardif ([18]), specialised to the present scope.

Definition 3.10. Let $\mathbb{T}$ be an elementary H-tree. An *incidence* function f on $\mathbb{T}$ is a map

$$f : T \to A(\mathbb{T}) \cup H$$

such that for each $v \in T$, $f(v)$ is an arc incident to v or a colour h such that $v \in S_h(\mathbb{T})$.

Let $\mathbb{D} = \mathbb{D}(\mathbb{T})$ be the H-digraph with constants defined as follows. Let the vertex set of $\mathbb{D}$ be the set of incidence functions of $\mathbb{T}$. For vertices f and g of $\mathbb{D}$, let (f,g) be an arc *unless* there exists an arc $a = (s,t)$ of $\mathbb{T}$ such that $f(s) = a = g(t)$. Let the vertex f be h-coloured *unless* there is some vertex $v \in T$ such that $f(v) = h$.

Theorem 3.11 ([18])**.** *Let $\mathbb{H}$ be a digraph with constants and $\mathbb{T}$ be a H-tree. Then $\{\mathbb{T}\}$ is a duality for $\mathbb{D}(\mathbb{T})$.*

The final result we will need is the following which was essentially shown in [13].

Lemma 3.12. *Let $\mathbb{T}$ be an elementary H-tree with $k-1$ leaves. Then the graph $\mathbb{D}(\mathbb{T})$ admits a $k-$NU polymorphism.*

Proof. It is easy to see that the so-called *degree of monstrosity* of $\mathbb{T}$ (see [13]) is at most $k-1$: indeed, the leaves of $\mathrm{Inc}(\mathbb{T})$ are the colours in $\mathrm{Block}(\mathbb{T})$ and on their removal the leaves are exactly the leaves of $\mathbb{T}$, of which there are $k-1$. It follows from Lemma 4.2 of [13] that the dual $\mathbb{D}(\mathbb{T})$ admits a $k-$NU polymorphism. □

4. Reflexive Duality

Though we are assuming that $\mathbb{H}$ is a reflexive digraph with constants, neither the trees in the duality we get from Corollary 3.7 in the case that $\mathbb{H}$ is $k-$NU, nor their duals defined in Definition 3.10, are reflexive. But they should be.

Definition 4.1. Given an H-digraph $\mathbb{G}$ let $\mathbb{G}^{+r}$ be the reflexive H-digraph we get from $\mathbb{G}$ by adding the arc (v,v) for each vertex v of $\mathbb{G}$, and let $\mathbb{G}^{-r}$ be the irreflexive H-digraph we get by removing any arcs (v,v). Let $\mathbb{G}^{u}$ be the subdigraph of $\mathbb{G}$ induced by the vertices v such that (v,v) is an arc.

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The following facts are not hard to show.

Fact 4.2. *Let $\mathbb{T}$, $\mathbb{H}$ and $\mathbb{G}$ be H-digraphs. If $\mathbb{H}$ is reflexive, then the following hold:*

(i) $\mathbb{T}^{-r} \to \mathbb{H} \iff \mathbb{T} \to \mathbb{H} \iff \mathbb{T}^{+r} \to \mathbb{H}$

(ii) $\mathbb{H} \to \mathbb{G} \iff \mathbb{H} \to \mathbb{G}^u$

Definition 4.3. A *reflexive duality* for a reflexive digraph $\mathbb{H}$ with constants is a family $\mathcal{D}$ of reflexive H-digraphs such that for any reflexive H-digraph $\mathbb{G}$, the following holds.

$$(\forall \mathbb{D} \in \mathcal{D}, \mathbb{D} \not\to \mathbb{G}) \iff \mathbb{G} \to \mathbb{H}.$$

Observe for a tree $\mathbb{T}$, $\mathbb{T}^{+r}$ is not technically a tree. We will refer to it as a *reflexive tree*. If $\mathbb{T}$ is an elementary tree, we call $\mathbb{T}^{+r}$ a *reflexive elementary tree*. Nor can $\mathbb{T}^{+r}$ be critical as removing loops does not change what it maps to; but we call it critical if $\mathbb{T}$ is.

As its proof is just arrowing arguments, Lemma 3.2 holds for reflexive dualities. Further, the following comes directly from Corollary 3.7

Corollary 4.4. *. A reflexive digraph with constants $\mathbb{H}$ is $k-$NU if and only if it has a reflexive duality of reflexive critical elementary H-trees, each having at most $(k-1)$ leaves.*

Proof. This follows from Corollary 3.7 by showing that $\mathcal{D}$ is a duality for $\mathbb{H}$ if and only if $\mathcal{D}^{+r}$ is a reflexive duality for $\mathbb{H}$, where $\mathcal{D}^{+r} = \{\mathbb{T}^{+r} \mid \mathbb{T} \in \mathcal{D}\}$.

Assume that $\mathcal{D}$ is a duality. To see that $\mathcal{D}^{+r}$ is a reflexive duality let $\mathbb{G}$ be a reflexive H-digraph. We have by item (i) of Fact 4.2

$$\begin{aligned} \mathbb{G} \to \mathbb{H} &\iff \\ \mathbb{G}^{-r} \to \mathbb{H} &\iff (\forall \mathbb{T}^{+r} \in \mathcal{D}^{+r}, \mathbb{T}^{+r} \not\to \mathbb{G}) \\ &\iff (\forall \mathbb{T} \in \mathcal{D}, \mathbb{T} \not\to \mathbb{G}). \end{aligned}$$

The reverse implication is essentially the same.

□

The following is just as immediate from Theorem 3.11.

Corollary 4.5. *Let $\mathbb{H}$ be digraph with constants and $\mathbb{T}$ be a reflexive H-tree. Then $\{\mathbb{T}\}$ is a reflexive duality for $\mathbb{D}^u(\mathbb{T}^{-r})$.*

5. Outline of finite duality using a simple example

For a reflexive digraph $\mathbb{H}$ with constants that admits a $k-$NU polymorphism, we have by Corollary 3.7 that there is a duality $\mathcal{T}_{\mathbb{H}}$ of critical H-trees for $\mathbb{H}$. In the paper [7], which we are extending, we had analogous results, but we could further assume that $\mathcal{T}_{\mathbb{H}}$ was finite. Using this, we were able to express $\mathbb{H}$ as the retract of a product of finitely many tree duals, which were defined in an analogue of Definition 3.10. In the case that $\mathbb{H}$ is a reflexive digraph, we cannot assume that $\mathcal{T}_{\mathbb{H}}$ is finite.

Consider, for example, the reflexive directed arc with constants $\mathbb{A}$ having vertices 0 and 1. So $\mathbb{A}$ has colour 0 on the vertex 0, colour 1 on the vertex 1, and the arcs $(0,0),(0,1)$, and $(1,1)$. For any $i \geq 1$, the path P_i of length i from a vertex with colour 1 to a vertex with colour 0 does not admit a colour preserving map to $\mathbb{A}$ (see

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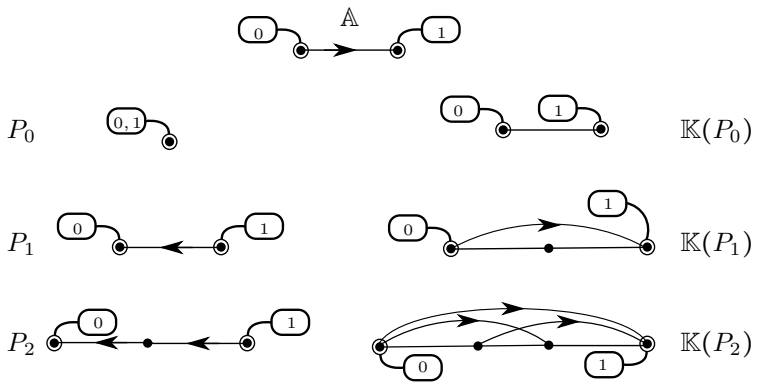

FIGURE 1. The arc $\mathbb{A}$, critical obstructions P_i, and their duals $\mathbb{K}(P_i)$ (Colours are shown in 'labels'; loops on vertices are not shown.)

Figure 1). So any duality for $\mathbb{A}$ must contain a tree that maps to such a path. It is not hard to see that

$$\mathfrak{T}_{\mathbb{A}} = \{P_0, P_1, P_2, P_3, \dots\}$$

is, in fact, a duality for $\mathbb{A}$.

Now, one of the points of moving to reflexive dualities is that doing so adds a lot of homomorphisms between trees of a duality $\mathfrak{T}_{\mathbb{H}}$, allowing us, by Lemma 3.2 to remove trees from $\mathfrak{T}_{\mathbb{H}}$ and retain the duality.

In $\mathfrak{T}_{\mathbb{A}}$, we have that $P_2 \to P_1$ so P_1 can be removed from $\mathfrak{T}_{\mathbb{A}}$. Indeed, we can remove any tree from $\mathfrak{T}_{\mathbb{A}}$, as long as we leave a tree above it; but leaving a tree above it prevents us from making $\mathfrak{T}_{\mathbb{A}}$ finite.

Clearly there is no single digraph, or even finite set of digraphs, that maps to all paths in $\mathfrak{T}_{\mathbb{A}}$. Our solution is to replace the whole chain $\mathfrak{T}_{\mathbb{A}}$ with a single limiting object P_∞ which does. This P_∞ is not strictly a digraph, but can be viewed as an infinite path P_∞ from a vertex with colour 1 to a vertex with colour 0.

In the general case of a digraph $\mathbb{H}$, we will partition the duality $\mathfrak{T}_{\mathbb{H}}$ up into finitely many 'chains' of trees each of which fits nicely together like $\mathfrak{T}_{\mathbb{A}}$ to define a limiting object. The limiting object is called an H_∞-tree, and is defined in Definition 6.1. The details of partitioning the duality $\mathfrak{T}_{\mathbb{H}}$ are given in Section 7.

We then, in Definition 6.2, generalise Nešetřil and Tardif's dual construction (Definition 3.10) to construct a so-called dual of an H_∞-tree, which admits as a duality the chain of trees that define it as a limiting object. With this, in Section 8 we get a representation of $\mathbb{H}$ as a retract of a product of finitely many duals of H_∞-trees.

We finish this overview by revealing the dual of the limiting object P_∞– a finite digraph for which $\mathfrak{T}_{\mathbb{A}}$ is a reflexive duality. This simple example can be done without appealing to Nešetřil and Tardif's dual construction, and infact, suggests the main idea in our generalisation of it. It turns out that $\mathbb{A}$ itself is the dual we need.

We will see in Section 6.1 that the digraph $\mathbb{K}_i = \mathbb{K}(P_i)$ on the vertices $\{0, 1, \dots, i+1\}$ with $(x, y) \in A(\mathbb{K}_i)$ if $x - y \leq 1$, (see Figure 1) is a reflexive dual for P_i. It follows from Theorem 3.11, Fact 4.2, and Lemma 3.2, that $\mathfrak{T}_{\mathbb{A}}$ is a duality for $\prod_{i=1}^{\infty} \mathbb{K}_i$.

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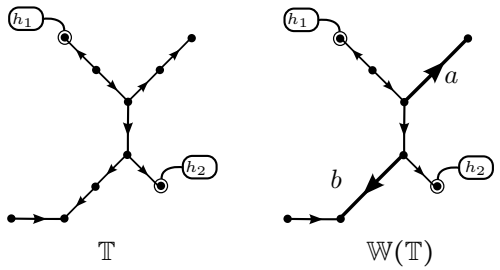

FIGURE 2. A tree $\mathbb{T}$ and its alternating template $\mathbb{W}(\mathbb{T})$.

The essential fact in our construction of the dual of P_∞ is that although $\prod_{i=1}^{N} \mathbb{K}_i$ has no retraction to $\mathbb{A}$ for any N, $\prod_{i=1}^{\infty} \mathbb{K}_i$ does. Indeed define $r : \prod_{i=1}^{\infty} \mathbb{K}_i \to \mathbb{A}$ by

$$r(x) = \begin{cases} 1 & \text{if } \limsup_{i\to\infty} x(i) = \infty \\ 0 & \text{otherwise.} \end{cases}$$

This is a homomorphism as $x \to y$ only if $x(i) - y(i) \leq 1$ for all i, so

$$r(x) = 1 \Rightarrow \limsup_{i\to\infty} x(i) = \infty \Rightarrow \limsup_{i\to\infty} y(i) = \infty \Rightarrow r(y) = 1.$$

So $\mathcal{T}_{\mathbb{A}}$ is a duality for $\mathbb{A} \trianglelefteq \prod_{i=1}^{\infty} \mathbb{K}_i$, as needed.

6. H_∞-trees and their duals

Throughout this section, $\mathbb{H}$ will always be a reflexive digraph with constants. In Subsection 6.1 we define the notion of an H_∞-tree ϕ and give some related definitions. In Subsection 6.2 we define its dual $\mathbb{K} = \mathbb{K}(\phi)$. In Subsection 6.3 we prove that $\mathbb{K}$ is the dual that we want, and that in the case that ϕ is a tree, that it is in fact the the reflexive Nešetřil-Tardif dual $\mathbb{D}^u(\phi^{-r})$.

6.1. Alternating Templates and H_∞-trees.

For an elementary reflexive H-tree $\mathbb{T}$, the *alternating template* $\mathbb{W} = \mathbb{W}(\mathbb{T})$ is the irreflexive H-tree we get from $\mathbb{T}^{-r}$ by replacing any directed path $u \to x_1 \to x_2 \to \cdots \to x_d \to v$ in which all vertices but u and v have degree 2, with the arc $u \to v$. That is; we successively contract one of the arcs incident to any non-sink and non-source vertex of degree 2. (See Figure 2.) As the leaves of $\mathbb{W}$ are exactly the leaves of $\mathbb{T}$, and $\mathbb{T}$ is elementary, the colours of $\mathbb{W}$ are exactly the colours of $\mathbb{T}$. A tree $\mathbb{W}$ is an *alternating template* if it is the alternating template of some elementary reflexive H-tree, $\mathbb{T}$.

The *expansion* of $\mathbb{T}$ over its alternating template $\mathbb{W} = \mathbb{W}(\mathbb{T})$ is the function $\phi = \phi_{\mathbb{T}} : A(\mathbb{W}) \to \{2, 3, 4, \dots\}$ that maps an arc $a = (u, v)$ to the number of vertices, including u and v, in the path in $\mathbb{T}$ that was replaced by a to make $\mathbb{W}$. It is clear that $\mathbb{T}$ is uniquely defined by the pair $(\mathbb{W}, \phi)$. The expansion of the tree $\mathbb{T}$ over the alternating template $\mathbb{W}(\mathbb{T})$ in Figure 2 is the function ϕ that takes the arcs a and b to 3, and all other arcs to 2.

The following definition is thus a generalisation of elementary reflexive H-trees.

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Definition 6.1. An H_∞-*tree* ϕ is an expansion function

$$\phi : A \to \{2, 3, \dots\} \cup \{\infty\}$$

defined on the arcset $A = A(\phi) = A(\mathbb{W})$ of an alternating template $\mathbb{W} = \mathbb{W}(\phi)$. Arcs $a \in A$ for which $\phi(a) = \infty$ are *infinite arcs*, other arcs are *finite arcs*. $A_\infty(\phi)$ is the set of infinite arcs and $A_F(\phi)$ is the set of finite arcs. If $A_\infty(\phi) = \varnothing$ then ϕ is a *finite* H_∞-*tree*; it is simply an elementary reflexive H-tree.

6.2. **Definition of the Duals $\mathbb{K}$ and $\mathbb{K}^0$.** Given an H_∞-tree ϕ, let $\mathbb{W}, A, A_\infty$ and A_F denote $\mathbb{W}(\phi), A(\phi), A_\infty(\phi)$ and $A_F(\phi)$ respectively, and let $m = |A|$. An arc $a = (x, y) \in A$ is an *in-arc* of y and an *out-arc* of x.

Definition 6.2. The *dual digraph* $\mathbb{K} = \mathbb{K}(\phi)$ is as follows.
Vertices: The vertices of $\mathbb{K}$ are the m-tuples $x = (x(1), \dots x(m))$ of

$$\prod_{a \in A_\infty(\phi)} \{0, \phi(a)\} \times \prod_{a \in A_F(\infty)} \{0, 1, \dots, \phi(a)\}$$

satisfying the following conditions.

- (V_0) For each non-leaf v of $\mathbb{W}$ there is at least one out-arc a of v with $x(a) = 0$ or one in-arc a of v with $x(a) = \phi(a)$, **and**
- (V_1) if an arc a in A_F satisfies (V_0) then no other arcs incident to v do.

Arcs: A pair (x, y) of vertices of $\mathbb{K}$ is an arc if for all $a \in \mathbb{W}$,

$$y(a) \le x(a) + 1.$$

Colours: A vertex x gets the colour h if some leaf of $\mathbb{W}$ has colour h, and for every leaf ℓ of $\mathbb{W}$ having colour h we have, where a is the arc incident to ℓ,

- a is an out-arc of ℓ and $x(a) = 0$, or
- a is an in-arc of ℓ and $x(a) = \phi(a)$.

See Figure 3 for examples.

Definition 6.3. The *loose dual* $\mathbb{K}^0(\phi)$ is the slightly larger digraph we get dropping vertex condition (V_1).

The reason for defining $\mathbb{K}^0$ is that it has a slightly easier definition which, with the following lemma, will make it easier to define maps to $\mathbb{K}$.

Definition 6.4. For a vertex x in $\mathbb{K}^0$ we say that an arc $a \in A$ of $\mathbb{W}$ that is incident to a non-leaf vertex v, is *good for v in x* if it satisfies (V_0) for v in x. That is, if $x(a) = 0$ and a is an out-arc of v, or $x(a) = \phi(a)$ and a is an in-arc of v.

So $\mathbb{K}$ is the subdigraph of $\mathbb{K}^0$ induced by vertices x that have a *unique* good arc $a \in A_F$ for each non-leaf vertex v of $\mathbb{W}$.

Lemma 6.5. *There is a retraction $r : \mathbb{K}^0(\phi) \to \mathbb{K}(\phi)$.*

Proof. Let $\mathbb{K}^0 = \mathbb{K}^0(\phi)$ and $\mathbb{K} = \mathbb{K}(\phi)$, and let the arcs $A = \{a_1, \dots, a_m\}$ of $\mathbb{W}$ be ordered so that the infinite arcs come first: $A_\infty = \{a_1, \dots, a_{|A_\infty|}\}$. Let $r : K^0 \to K : x \mapsto rx$ be the map where the vertex rx of $\mathbb{K}$ is defined as follows. For $a_i \in A$ let $rx(a_i) = x(a_i)$ unless a_i is a finite arc that is good for some v in x and there is another arc a_j with $j < i$ that is also good for v in x. In this case let

(i) $rx(a_i) = x(a_i) + 1 = 1$ if a_i is an out-arc of v, or
(ii) $rx(a_i) = x(a_i) - 1 = \phi(a_i) - 1$ if a_i is an in-arc of v.

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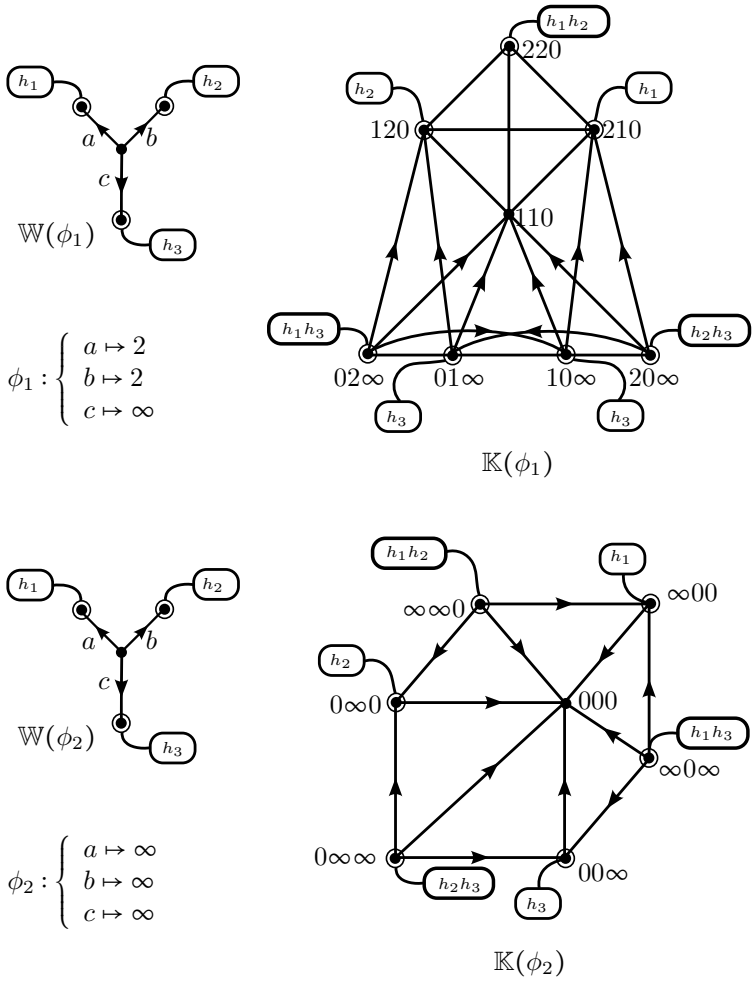

FIGURE 3. Two H_∞-trees ϕ_i and their duals $\mathbb{K}(\phi_i)$. We write 02∞ for the vertex $(0, 2, \infty)$ (and each triple denotes the tuple $(\phi_i(a), \phi_i(b), \phi_i(c))$).

Now, it is clear that r maps K^0 to K, and is the identity on K. Indeed, the vertices x of $\mathbb{K}^0$ that are not fixed by r are exactly those that do not satisfy condition (V_1); and the definition of rx in this case ensures that rx does satisfy it. What remains to be shown is that $x \mapsto rx$ is a homomorphism.

To show that $x \to y$ implies $rx \to ry$, it is enough to show that for every $a \in A$,

$$y(a) \le x(a) + 1 \implies ry(a) \le rx(a) + 1.$$

As r can change a coordinate by at most 1, this is trivial unless $ry(a) = y(a) + 1$, in which case $y(a) = 0$ and so $ry(a) = 1$, or unless $rx(a) = x(a) - 1$, in which case $rx(a) = \phi(a) - 1$. Either way, $ry(a) \le rx(a) + 1$ is satisfied.

To see that r preserves colours, notice that if x has colour h, and a is an arc incident to a leaf with colour h, then a cannot be good for any non-leaf vertex so

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$rx(a) = x(a)$; since having colour h depends only on the values on these arcs, rx also has colour h.

□

6.3. **Proof of Duality.** In this section, we prove Theorem 6.15 which essentially says that $\mathbb{K}(\phi)$ really is the 'dual' we are looking for. In the case that ϕ is a finite tree, this is immediate by Lemma 6.7 which shows that $\mathbb{K}$ is a reflexive version of the Nešetřil-Tardif dual $\mathbb{D}$ from Definition 3.10. For the full theorem we also need to prove that $\mathbb{K}(\phi)$ is a 'limit' of the duals of finite H_∞-trees. To be more concrete, we need a definition.

Definition 6.6. For an infinite tree ϕ, the *N-realisation* of ϕ is the finite H_∞-tree $\mathbb{T}^\phi_N$ with expansion ϕ_N over the same alternating template $\mathbb{W}(\phi)$, where $\phi_N(a) = N$ for $a \in A_\infty(\phi)$ and $\phi_N(a) = \phi(a)$ for $a \in A_F(\phi)$.

In Lemma 6.9 we make the connection between our dual of the "infinite tree" and the duals of its N-realisations by proving that

$$\mathbb{K}(\phi) \trianglelefteq \prod_{N \geq N_0} \mathbb{K}^0(\mathbb{T}^\phi_N).$$

After we prove the lemmas, Theorem 6.15 is immediate.

The proof of the following is similar to that of an analogous result in [7]. Recall that a finite H_∞-tree is just a reflexive elementary H-tree.

Lemma 6.7. *For a finite H_∞-tree $\mathbb{T}$, $\mathbb{K}(\mathbb{T})$ is isomorphic to $\mathbb{D}^u = \mathbb{D}^u(\mathbb{T}^{-r})$.*

Proof. We define a vertex map

$$V(\mathbb{D}^u) \to V(\mathbb{K}) : f \mapsto x_f,$$

and then show that it is an isomorphism.

For an arc $a \in A = A(\phi_{\mathbb{T}})$ of the template $\mathbb{W} = \mathbb{W}(\mathbb{T})$ let $v^a_0 \to v^a_1 \to \cdots \to v^a_{\phi(a)-1}$ be the directed path in $\mathbb{T}$ that contracts to a. For a vertex f of $\mathbb{D}^u$, (which is a function from the vertices of $\mathbb{T}$ to its arcs and labels) we say that f maps v^a_i *up (on a)* for $i \in [0, \phi(a) - 2]$ if $f(v^a_i) = (v^a_i, v^a_{i+1})$, and it maps v^a_i *down on a* for $i \in [1, \phi(a) - 1]$ if $f(v^a_i) = (v^a_{i-1}, v^a_i)$. Also f maps $v^a_{\phi(a)-1}$ up on a if it does not map it down on a, and maps v^a_0 down on a if it does not map it up on a. Notice that, since an element f of $\mathbb{D}^u(\mathbb{T}^{-r})$ is a loop, if it maps v^a_i up on a then it must map v^a_j up on a for all $j \geq i$.

To define x_f, let $x_f(a)$ be the minimum $i \in [0, \phi(a) - 1]$ such that f maps v_i up on a, if such an i exists, and otherwise (if f maps all v_i down) let $x_f(a) = \phi(a)$.

We must now show that this map $f \mapsto x_f$ is an isomorphism. To see that it is a bijection, we define its inverse, $x \mapsto f_x$ as follows. For a vertex $v \in T$ having degree at least 3, v is a non-leaf vertex of $\mathbb{W}$, so by condition V_1 there is exactly one incident arc $a \in A$ that satisfies V_0 for v in x. Let $f_x(v)$ be the arc in the path contracting to a that is incident to v. For a vertex $v \in T$ of degree 1 or 2, $v = v^a_i$ for a unique arc a of A and unique i. Let $f_x(v^a_i)$ map up on a if $x(a) \leq i$ and down on a if $i < x(a)$.

It is easy to see that these maps are inverses of one another. Indeed, for $a \in \mathbb{W}$ we have that $x_{f_x}(a)$ is the minimum i such that f_x maps v^a_i up on a; but f_x maps v^a_i up for all $i \geq x(a)$. So $x_{f_x}(a) = x(a)$. On the other hand, for $v = v^a_i \in T$, f_{x_f} maps v up on a if $x_f(a) \leq i$, which is true only for those i on which f maps v up on a.

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To see that both maps preserve arcs: recall that (f,g) is not an arc in $\mathbb{D}^u$ if there is some arc (v,v') of $\mathbb{T}$ such that $f(v) = (v,v') = g(v')$. We can write $v = v_i^a$ and $v' = v_{i+1}^a$ for some arc a of A and some i, so we have $f(v_i^a) = (v,v') = g(v_{i+1}^a)$. But then f maps v_i^a up, so $x_f(a) \le i$, and g maps v_{i+1}^a down, so $x_g(a) > i+1$. But then $x_g(a) > x_f(a)+1$, ensuring that (x_f,x_g) is not an arc in $\mathbb{K}$. Conversely, if (x_f,x_g) is not an arc, then there exists some a such that $x_g(a) > x_f(a)+1$. Let $x_f(a) = i$. Then clearly $i < \phi(a) - 1$ and f maps v_i^a up on a, and $x_g(a) > x_f(a)+1$ implies that g cannot map v_{i+1}^a up, hence by definition of x_f and x_g we have that $f(v_i^a) = (v_i^a, v_{i+1}^a) = g(v_{i+1}^a)$ and so (f,g) is not an arc.

To see that the maps preserve colours, recall that f is coloured h if and only if there is no leaf ℓ such that $f(\ell) = h$, i.e. for any leaf ℓ in $\mathbb{T}$ with colour h, we have f mapping ℓ up its incident out-arc a, or down its incident in-arc a. This is equivalent to saying that $x_f(a)$ is 0 or $\phi(a)$ respectively, i.e. that x_f gets colour h. □

By Corollary 4.5 and Lemma 3.2 we get that $\{\mathbb{T}^{-r}\}$ is a duality for $\mathbb{K} = \mathbb{K}(\mathbb{T})$, and so $\{\mathbb{T}\}$ is a reflexive duality for $\mathbb{K}$. As $\mathbb{K} \trianglelefteq \mathbb{K}^0$, we have that $\mathbb{K}$ and $\mathbb{K}^0$ are homomorphically equivalent, we get the following.

Corollary 6.8. *For a reflexive $\tau_{\mathbb{H}}$-tree $\mathbb{T}$, $\{\mathbb{T}\}$ is a reflexive duality for $\mathbb{K}^0(\mathbb{T})$.*

The following is the technical heart of the paper.

Lemma 6.9. *Let ϕ be an H_∞-tree and $N_0 \ge 2$ be fixed. Then*

$$\mathbb{K}(\phi) \trianglelefteq \prod_{N \ge N_0} \mathbb{K}^0(\mathbb{T}_N^\phi).$$

Proof. All products will be over the index set $N \ge N_0$ so we write $\prod$ for $\prod_{N \ge N_0}$. Similarly we write such things as $\liminf$ for $\liminf_{N\to\infty}$. By Lemma 6.5 it is enough to show that $\mathbb{K}^0(\phi) \trianglelefteq \prod \mathbb{K}^0(\mathbb{T}_N^\phi)$.

First we define the retraction $r : \prod \mathbb{K}^0(\mathbb{T}_N^\phi) \to \mathbb{K}^0(\phi)$. Choose an arbitrary root vertex r_0 of $\mathbb{W} = \mathbb{W}(\phi)$, and for $x = (x_{N_0}, x_{N_0+1}, \dots,) \in \prod \mathbb{K}^0(\mathbb{T}_N^\phi)$, let $r(x) = rx$ be defined by setting $rx(a)$ as follows for $a \in A$.

If a is oriented towards r_0 let

$$rx(a) = \begin{cases} \liminf x_N(a) & \text{if } a \in A_F(\phi) \\ 0 & \text{if } a \in A_\infty(\phi) \text{ and } \liminf x_N(a) < \infty \\ \infty & \text{if } a \in A_\infty(\phi) \text{ and } \liminf x_N(a) = \infty. \end{cases}$$

If a is oriented away from r_0 let

$$rx(a) = \begin{cases} \limsup x_N(a) & \text{if } a \in A_F(\phi) \\ \infty & \text{if } a \in A_\infty(\phi) \text{ and } \liminf(N - x_N(a)) < \infty \\ 0 & \text{if } a \in A_\infty(\phi) \text{ and } \liminf(N - x_N(a)) = \infty. \end{cases}$$

Observe that when a is oriented towards r_0 we get $rx(a) \ne 0$ implies $x_N(a) > 0$ for all but finitely many values of N. When a is oriented away from r_0 we get that $rx(a) \ne \phi(a)$ implies $x_N(a) < N$ for all but finitely many values of N. This yields the following, which is the point of the somewhat fiddly definition of rx.

Claim 6.10. *Let v be a vertex of $\mathbb{W}$. For any arc a incident to v, either a is good for v in rx or else there are only finitely many N for which a is good for v in x_N.*

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Proof. For a vertex v let a be the incident arc that is on a path between it and r_0. First consider the case that a is oriented towards r_0, so is an out-arc of v. Assume that a is not good for v in rx. As a is an out-arc of v, this means by Definition 6.4 that $rx(a) \neq 0$. By the above observation this implies that $x_N(a) > 0$ for all but finitely many values of N, and so a is not good for all but finitely many of the vertices x_N.

A similar argument can be made in the case that a is oriented away from r_0: 'towards' becomes 'away from', 'out-arc' becomes 'in-arc', '$rx(a) \neq 0$' becomes '$rx(a) \neq \phi(a)$' and '$x_N(a) > 0$' becomes '$x_N(a) < N$'.

Now let a be another arc incident to v, and consider the case that a is oriented towards r_0, so is an in-arc of v. Assuming that a is good for v in infinitely many of the x_N, meaning that $x_N(a) = N$ for infinitely many N, we get that $\liminf x_N(a) = \infty$, so $rx(a) = \infty$. Thus a is good for v in rx.

A similar argument can be made in the case that a is oriented away from r_0. ⋄

We are ready to show that r is a retraction.

Claim 6.11. *r maps to $\mathbb{K}^0(\phi)$.*

Proof. Fix a vertex v of $\mathbb{W}$. Since it has finite degree, there is an arc a incident to v that is good for v in x_N for infinitely many N. By Claim 6.10, this means a is good for v in rx.

⋄

Claim 6.12. *r is an H-homomorphism.*

Proof. To see that r preserves colours, let x have colour h. Then for any arc $a \in A$ with a leaf ℓ having colour h we have that either a is an out-arc of ℓ and $x_N(a) = 0$ for all $N \geq N_0$, or a is an in-arc of ℓ and $x_N(a) = \phi(a)$ for all N. But then $rx(a) = 0$ or $\phi(a)$ respectively, so rx has colour h.

To see that r preserves arcs, let $x \to y$, so for all $a \in A$ and $N \geq N_0$, $y_N(a) \leq x_N(a) + 1$. We show that $rx \to ry$ by showing that for all a, $ry(a) \leq rx(a) + 1$.

Taking liminfs of both sides of $y_N(a) \leq x_N(a) + 1$ we get that

$$\liminf y_N(a) \leq \liminf x_N(a) + 1.$$

In the case that a is oriented towards r_0, this means that $ry(a) \leq rx(a) + 1$ in the case that $a \in A_F(\phi)$ and $ry(a) = rx(a)$ in the case that $a \in A_\infty(\phi)$.

If a is oriented away from r_0 then we take limsups of $y_N(a) \leq x_N(a) + 1$ for the case that $a \in A_F(\phi)$. When $a \in A_\infty(\phi)$ we observe that $N - y_N(a) \geq N - x_N(a) - 1$ and take liminfs of both sides of this.

⋄

Let $c : \mathbb{K}^0(\phi) \to \prod \mathbb{K}^0(\mathbb{T}^\phi_N)$ be defined by letting $c(x) = cx = (cx_{N_0}, cx_{N_0+1}, \dots)$, where for each $a \in A$

$$cx_N(a) = \begin{cases} N & \text{if } x(a) = \infty \\ x(a) & \text{otherwise.} \end{cases}$$

It is easy to verify that c does indeed map to $\prod \mathbb{K}^0(\mathbb{T}^\phi_N)$.

Claim 6.13. *c is an H-homomorphism.*

Proof. To see that it preserves colours, assume that cx does not have the colour h. So there is some arc $a \in A$ with a leaf ℓ having colour h, such that either a is an out-arc of ℓ and $cx(a) > 0$, or a is an in-arc and $cx(a) < \phi(a)$. In the first case, we

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have that $x(a) > 0$, and in the second case $x(a) < \phi(a)$. Either way, c does not get colour h. Now let (x, y) be an arc, i.e. $y(a) \leq x(a) + 1$ for all a. We must show that $cy_N(a) \leq cx_N(a) + 1$ for all a and all $N \geq N_0$. This is clear if both $x(a)$ and $y(a)$ are finite; otherwise we certainly have that $x(a) = \infty$, and then $y(a) \in \{0, \infty\}$ and the inequailty is verified in both cases.

$\diamond$

The following claim shows that r is a retraction, and so finishes our proof.

Claim 6.14. *$r \circ c$ is the identity on $\mathbb{K}^0(\phi)$.*

Proof. Let $x \in \mathbb{K}^0(\phi)$. We must show for all arcs $a \in A$ that $x(a) = rcx(a)$.

If $x(a) = \infty$ then $cx_N(a) = \phi_N(a) = N$ for all N, and all $a \in A_\infty(\phi)$. If a is oriented towards r_0 then $\liminf cx_N(a) = \liminf N = \infty$ gives us that that $rcx(a) = \infty$. When a is oriented away from r_0 then $\liminf(N - cx_N(a)) = \liminf 0 = 0$ gives us that that $rcx(a) = \infty$.

So we may assume that $x(a) = n$ for some integer $n < \infty$. If $a \in A_\infty(\phi)$ then $x(a)$ can only be 0, so $cx_N(a) = 0$ for all N, and so whichever way a is oriented, $rcx(a) = 0$, as needed. In the case that $a \in A_F(\phi)$, $x(a) = n$ means that $cx_N(a) = n$ for all n and so $rcx(a) = n$. $\diamond$

$\square$

This completes the proof of the two lemmas, and now we prove Theorem 6.15.

Theorem 6.15 (Duality for H_∞-trees)**.** *For any H_∞-tree ϕ, and any integer $N_0 \geq 2$, the family $\Phi_{N_0} = \{\mathbb{T}_N^\phi \mid N \geq N_0\}$ of N-realisations of ϕ is a reflexive duality for $\mathbb{K}(\phi)$.*

Proof. By Lemma 6.7 $\{\mathbb{T}_N^\phi\}$ is a reflexive duality for $\mathbb{K}^0(\mathbb{T}_N^\phi)$, so by Lemma 3.2 (i) (which recall holds for reflexive dualities also), Φ_{N_0} is a reflexive duality for $\prod_{N \geq N_0} \mathbb{K}^0(\mathbb{T}_N^\phi)$. By Lemma 3.2 (ii) and Lemma 6.9 we then have that it is a reflexive duality for $\mathbb{K}(\phi)$.

$\square$

6.4. The $k-\mathrm{NU}$ operation on $\mathbb{K}(\phi)$.

Lemma 6.16. *Let ϕ be an H_∞-tree with $k-1$ leaves. The digraph $\mathbb{K}(\phi)$ admits a $k-\mathrm{NU}$ polymorphism.*

Proof. By Lemma 3.12 the digraph $\mathbb{D}((\mathbb{T}_N^\phi)^{-r})$ admits a $k-\mathrm{NU}$ polymorphism for each $\mathbb{T}_N^\phi \in \Phi_{N_0}$. As a $k-\mathrm{NU}$ operation must preserve loops, and we proved in the proof of Lemma 6.7 that $\mathbb{K}(\mathbb{T}_N^\phi) = \mathbb{D}^u((\mathbb{T}_N^\phi)^{-r})$, we have that $\mathbb{K}(\mathbb{T}_N^\phi)$ has a $k-\mathrm{NU}$ for all $\mathbb{T}_N^\phi$. So by Lemma 2.1, $\prod_{N \geq N_0} \mathbb{K}(\mathbb{T}_N^\phi)$ does, and by Lemmas 2.2 and 6.9, $\mathbb{K}(\phi)$ does too. $\square$

In fact, we can give a $k-\mathrm{NU}$ polymorphism on $\mathbb{K}^0(\phi)$ explicitly, and so using the retraction r_0 get one on $\mathbb{K}(\phi)$. Let $a_1, \ldots, a_m$ denote the arcs of $\mathbb{W} = \mathbb{W}(\phi)$. For each $1 \leq i \leq m$ define an integer c_i as follows: Remove the arc $a_i = (u_i, v_i)$ from $\mathbb{W}$ to obtain two connected components; let c_i be the number of leaves of $\mathbb{W}$ in the component containing u_i. Clearly $1 \leq c_i \leq k-2$. Viewing elements of $\mathbb{K}^0$ as columns for convenience of notation, define $f : (\mathbb{K}^0)^k \to \mathbb{K}^0$ by

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$$f\left(\begin{bmatrix} x_{1,1} \\ \vdots \\ x_{m,1} \end{bmatrix} \cdots \begin{bmatrix} x_{1,k} \\ \vdots \\ x_{m,k} \end{bmatrix}\right) = \begin{bmatrix} f_1(x_{1,1},\ldots,x_{1,k}) \\ \vdots \\ f_m(x_{m,1},\ldots,x_{m,k}) \end{bmatrix}$$

where f_i returns the (c_i+1)-th smallest entry in row i, i.e. if $\{x_{i,1},\ldots,x_{i,k}\} = \{u_1,\ldots,u_k\}$ where $u_i \le u_j$ when $i \le j$ then $f_i(x_{i,1},\ldots,x_{i,k}) = u_{c_i+1}$.

Fact 6.17. *The above map is a $k-$NU polymorphism of $\mathbb{K}^0$.*

We do not prove this, as we already have existence of a $k-$NU polymorphism on $\mathbb{K}^0$, but the proof is an immediate generalisation of a proof from [7].

7. Finiteness of the H_∞-tree Duality

For a reflexive digraph $\mathbb{H}$ with constants which admits a $k-$NU polymorphism, Corollary 4.4 gives us that there is a reflexive duality of critical trees for $\mathbb{H}$. Let $\mathfrak{T}_{\mathbb{H}}$ be the largest such duality.

We define a partial order on the set of all H_∞-trees as follows. Let $\phi \le \phi'$ if $\mathbb{W}(\phi) = \mathbb{W}(\phi')$ and $\phi(a) \le \phi'(a)$ for all $a \in A(\mathbb{W}(\phi))$. As H_∞-trees are reflexive, it is clear that if $\phi' \ge \phi$ then $\phi' \to \phi$; but the inverse statement is not true unless $\mathbb{W}(\phi') = \mathbb{W}(\phi)$.

The N-realisation $\mathbb{T}^\phi_N$ of ϕ was defined in Definition 6.6. An *N-witness* of ϕ is any finite H_∞-tree $\mathbb{T}$ such that $\mathbb{T}^\phi_N \le \mathbb{T} \le \phi$.

Lemma 7.1. *Let ϕ be an H_∞-tree. If $\mathfrak{T}_{\mathbb{H}}$ contains an $|H|$-witness for ϕ, then it contains every $|H|$-witness for ϕ.*

Proof. There is a directed walk of length $|H|$ from a vertex u of $\mathbb{H}$ to vertex v of $\mathbb{H}$ if and only if there is a directed walk of every length $N > |H|$ from u to v. It is easy to deduce from this that an $|H|$-witness of ϕ is a (critical) obstruction for $\mathbb{H}$ if and only if all $|H|$-witnesses are (critical) obstructions for $\mathbb{H}$. □

Let $\mathfrak{T}^\infty_{\mathbb{H}}$ be the family of H_∞-trees we get from $\mathfrak{T}_{\mathbb{H}}$ as follows: for any H_∞-tree ϕ for which there is an $|H|$-witness of ϕ in $\mathfrak{T}_{\mathbb{H}}$ add ϕ and remove every $|H|$-witness of ϕ. Let $\mathcal{M}^\infty_{\mathbb{H}}$ be the family of maximal elements of $\mathfrak{T}^\infty_{\mathbb{H}}$ with respect to our ordering of H_∞-trees.

Lemma 7.2. *If $\mathfrak{T}_{\mathbb{H}}$ is a reflexive duality for $\mathbb{H}$, then it is a reflexive duality for $\prod_{\phi\in\mathcal{M}^\infty_{\mathbb{H}}} \mathbb{K}(\phi)$.*

Proof. Let $\mathfrak{T}^{\mathrm{reg}}_{\mathbb{H}}$ be the family we get from $\mathfrak{T}_{\mathbb{H}}$ as follows. For any H_∞-tree ϕ for which $\mathfrak{T}_{\mathbb{H}}$ contains an $|H|$-witness of ϕ, we have by Lemma 7.1 that $\mathfrak{T}_{\mathbb{H}}$ contains all $|H|$-witness of ϕ; remove all but the N-realisation $\mathbb{T}^\phi_N$ for all $N \ge |H|$.

For any $|H|$-witness $\mathbb{T}$ of ϕ there is clearly some $N \ge |H|$ such that $\mathbb{T}^\phi_N$ maps to $\mathbb{T}$. So by Lemma 3.2 (iii) it is enough to show that $\mathfrak{T}^{\mathrm{reg}}_{\mathbb{H}}$ is a reflexive duality for $\prod_{\phi\in\mathcal{M}^\infty_{\mathbb{H}}} \mathbb{K}(\phi)$.

But $\mathfrak{T}^{\mathrm{reg}}_{\mathbb{H}}$ is just the family we get from $\mathcal{M}^\infty_{\mathbb{H}}$ by replacing each H_∞-tree ϕ with $\Phi_{|H|}$, so by Theorem 6.15 and Lemma 3.2 (i), this is true. □

Theorem 7.3. *For any reflexive $k-$NU digraph $\mathbb{H}$ with constants, the family $\mathcal{M}^\infty_{\mathbb{H}}$ of maximal elements of $\mathfrak{T}^\infty_{\mathbb{H}}$ is finite.*

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Proof. The poset of H_∞-trees with a given template $\mathbb{W}$ is clearly isomorphic to the product of $|A(\mathbb{W})|$ copies of the chain $\{2,3,\dots\}\cup\{\infty\}$. Since this chain is well-quasi-ordered, i.e. for every infinite sequence $a_1, a_2, \dots$ there exist indices $i<j$ such that $a_i \le a_j$, the same holds for the product [15]. In particular, the poset of H_∞-trees over $\mathbb{W}$ contains only finite antichains, so its intersection with $\mathcal{M}_{\mathbb{H}}^{\infty}$ is finite. Hence it is enough to show that the set of alternating templates of trees in $\mathcal{T}_{\mathbb{H}}$ is finite. Hence the proof follows from Lemma 7.4 below. □

Because we shall require it in the next section, we now prove a stronger result than what is needed for Theorem 7.3. The *diameter* of a reflexive digraph $\mathbb{H}$ is the minimum d such that there is a path of length d between any two vertices of the symmetrisation $\mathbb{H}^u$ of $\mathbb{H}$.

Lemma 7.4. *Let D be an integer. The set of alternating templates of trees in $\bigcup_{i\in I}\mathcal{T}_{\mathbb{H}_i}$, where the union is over all reflexive $k-$NU digraphs of diameter at most D, is finite.*

Proof. For an alternating template $\mathbb{W}$, the *symmetric template* $\mathbb{B}=\mathbb{B}(\mathbb{W})$ is the symmetric H-graph we get from the symmetrisation of $\mathbb{W}$ by replacing any path $u\sim x_1\sim\cdots\sim x_d\sim v$ in which all vertices but u and v have degree two, by the edge uv. $\mathbb{B}$ is a symmetric template of a tree $\mathbb{T}$ if it is the symmetric template of $\mathbb{W}(\mathbb{T})$. First we observe that there are only finitely many symmetric templates of trees in $\bigcup_{i\in I}\mathcal{T}_{\mathbb{H}_i}$.

Claim 7.5. *The set of symmetric templates of trees with at most $k-1$ leaves is finite.*

Proof. Indeed, the symmetric templates are trees with at most $k-1$ leaves and having no vertices of degree 2. Such trees can have at most $2k-4$ vertices; indeed, a tree on n vertices satisfies $\sum\deg(v)=2E=2n-2$, while if there are $k-1$ leaves and no degree two vertices we have $\sum\deg(v)\ge k-1+3(n-k+1)=3n-2k+2$. So there are finitely many symmetric templates, and finitely many choices of the colours from H on their leaves. ◇

The *expansion* $\alpha_{\mathbb{W}}$ of $\mathbb{W}$ (over $\mathbb{B}=\mathbb{B}(\mathbb{W})$) is the function that maps an edge $e\in\mathbb{B}$ to the number of arcs in the path in $\mathbb{W}$ replaced by e to make $\mathbb{B}$.

Claim 7.6. *For i in I and $\mathbb{T}$ in $\mathcal{T}_{\mathbb{H}_i}$ let $\mathbb{W}=\mathbb{W}(\mathbb{T})$. Then $\alpha_{\mathbb{W}}(e)\le 2D$ for each $e\in\mathbb{B}=\mathbb{B}(\mathbb{T})$.*

Proof. Let $e=uv$ be an edge of $\mathbb{B}$, and let e' be any arc on the path between u and v in $\mathbb{W}$. As $\mathbb{T}$ is critical there is a homomorphism $f:\mathbb{T}\setminus\{e'\}\to\mathbb{H}_i$. Now $f(u)$ and $f(v)$ have distance at most D in $\mathbb{H}_i^u$ so any alternating path of length $2D$ admits a homomorphism to $\mathbb{H}_i$ with one endpoint going to each of $f(u)$ and $f(v)$. As $\mathbb{T}\not\to\mathbb{H}_i$, we must therefore have that $\alpha_{\mathbb{W}}(e)<2D$. ◇

Clearly only finitely many symmetric templates $\mathbb{W}$ (in fact $2^{|E(\mathbb{B})|}$) can have the same expansion over $\mathbb{B}$, so from this claim we get that each of the finitely many symmetric templates of trees in $\bigcup_{i\in I}\mathcal{T}_{\mathbb{H}_i}$ yield only finitely many alternating templates of trees in $\bigcup_{i\in I}\mathcal{T}_{\mathbb{H}_i}$. This is what we needed to show. □

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8. Finite presentation of reflexive $k-$NU digraphs

Theorem 8.1. *A reflexive digraph $\mathbb{H}$ with constants is $k-$NU if and only if there is a finite family $\mathcal{M}$ of elementary H_∞-trees, each having at most $k-1$ leaves, such that $\mathbb{H} \trianglelefteq \prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$. If such a family exists, it can be assumed to be $\mathcal{M}^\infty_{\mathbb{H}}$.*

Further, if no two vertices of $\mathbb{H}$ have the same (closed) neighbourhoods, then $\mathcal{M}$ can be assumed to contain no trivial trees.

Proof. If $\mathbb{H} \trianglelefteq \prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$ then $\mathbb{H}$ is $k-$NU by Lemmas 6.16 and 2.2. For the other direction, assume that $\mathbb{H}$ is $k-$NU. By Corollary 4.4 $\mathbb{H}$ has a reflexive duality $\mathfrak{T}_{\mathbb{H}}$ of critical H-tree obstructions with at most $k-1$ leaves each. By Lemma 7.2 $\mathfrak{T}_{\mathbb{H}}$ is a reflexive duality for $\prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$ where $\mathcal{M} = \mathcal{M}^\infty_{\mathbb{H}}$, and $\mathcal{M}$ is finite by Theorem 7.3. It follows that $\mathbb{H}$ and $\prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$ are homomorphically equivalent. As $\mathbb{H}$ is a digraph with constants, it is a core, so $\mathbb{H} \trianglelefteq \prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$, as needed.

For the further bit, we simply observe that the underlying digraph dual of a trivial tree is a 2 vertex reflexive symmetric clique K_2. If a retract $\mathbb{H}$ of $\mathbb{G}\times K_2$ is not a retract of $\mathbb{G}$ then $\mathbb{H}$ contains vertices v and v' with the same image under the projection $\pi : \mathbb{G}\times K_2 \to \mathbb{G}$. So v and v' have the same neighbourhoods. □

We can also remove constants from Theorem 8.1. An *infinite tree* ϕ is an H_∞-tree ϕ' with colours removed. Its dual $\mathbb{K}(\phi)$ is the reflexive digraph we get from $\mathbb{K}(\phi')$ by removing colours.

Corollary 8.2. *A reflexive digraph $\mathbb{H}$ is $k-$NU if and only if there is a finite set $\mathcal{M}$ of infinite trees, each with at most $k-1$ leaves, such that $\mathbb{H} \trianglelefteq \prod_{\phi\in\mathcal{M}} \mathbb{K}(\phi)$.*

Proof. By Fact 2.4 and Lemma 6.16 the graph $\mathbb{K}(\phi)$ for an infinite tree ϕ with $k-1$ leaves has an $k-$NU polymorphism, so by Lemmas 2.1 and 2.2, $\mathbb{H} \trianglelefteq \prod_{\mathcal{M}} \mathbb{K}(\phi)$ does also.

On the other hand, assume that $\mathbb{H}$ is $k-$NU. Then by Fact 2.4, so is $\mathbb{H}^{+c}$, and so $\mathbb{H}^{+c} \trianglelefteq \prod_{\mathcal{M}} \mathbb{K}(\phi)$ for some family $\mathcal{M}$ of H_∞-trees. Clearly then $\mathbb{H} \trianglelefteq (\prod_{\mathcal{M}} \mathbb{K}(\phi))^{-c} = \prod_{\mathcal{M}} \mathbb{K}(\phi^{-c})$. □

As diameter cannot increase under taking product and retracts, there can be no finite set of generators for the variety of reflexive $k-$NU digraphs, but we can make the set finite if we bound their diameter. This was the reason for going the extra step and proving Lemma 7.4. It yields the following.

Theorem 8.3. *Let D and $k \geq 3$ be positive integers. Then there exists a finite set $\mathcal{G}_{D,k}$ of finite $k-$NU digraphs, (duals of infinite trees with at most $k-1$ leaves), such that any reflexive $k-$NU digraph of diameter at most D is is a retract of a finite product of digraphs in $\mathcal{G}_{D,k}$.*

9. Finite Duality

It follows from the results of [11] and [12] (see also [7]) that a reflexive graph $\mathbb{H}$ is NU if and only if $\mathbb{H}^{+c}$ has finite duality. We prove here that this result is actually valid for all strongly connected reflexive digraphs; in fact we characterise the digraphs with finite duality as those that are NU and strongly connected[1].

[1]Note that an alternative (and rather more involved) proof can be given using results in [14] and [11]; the only result from [14] we require here is Theorem 3.6, which we invoked in the proof of Corollary 4.4.

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Theorem 9.1. *Let $\mathbb{H}$ be a reflexive digraph. Then the following conditions are equivalent:*

(i) $\mathbb{H}^{+c}$ has finite duality.
(ii) $\mathbb{H}$ is strongly connected and admits an NU *polymorphism.*

Proof. On the one hand, assume that $\mathbb{H}^{+c}$ has finite duality. By Theorem 2.5 and Corollary 4.5 of [11], it admits an NU polymorphism, and so by Fact 2.4, $\mathbb{H}$ does too. If $\mathbb{H}$ is not strongly connected, then it has vertices u and v with no directed path from u to v; thus, any directed path with endpoints coloured u and v respectively is a critical obstruction of $\mathbb{H}$, contradicting the finite duality.

On the other hand, assume that $\mathbb{H}$ is strongly connected and admits an NU polymorphism. As $\mathbb{H}$ is strongly connected, there is a directed $|H|$-path between any two vertices. Thus no $|H|$-realisation of any H_∞-tree can be critical for $\mathbb{H}$, and so by Lemma 7.1 no $|H|$-witness can be. By Lemma 7.4, the duality $\mathfrak{T}_\mathbb{H}$ for $\mathbb{H}$, which exists by Corollary 4.4, has only finitely many alternating templates, and as $\mathfrak{T}_\mathbb{H}$ has no $|H|$-witnesses, only finitely many trees in $\mathfrak{T}_\mathbb{H}$ have the same alternating template. So $\mathfrak{T}_\mathbb{H}$ is finite, as needed. □

As [11] gives a very simple polynomial time algorithm for deciding if a structure has finite duality, this gives a simple algorithm for deciding if a strongly connected reflexive digraph is NU. A polynomial time algorithm for deciding if any reflexive digraph is NU is given in [14], but it is considerably more complicated.

10. Posets

In this section we restrict our attention to *posets* – acyclic, transitive reflexive digraphs. We start with a brief overview of relevant results about posets admitting NU polymorphisms.

Early results of [16] and [9] showed that the variety of $3-$NU posets is generated by *fences*, which are, in digraph terms, paths consisting of alternating arcs. In [20] the authors looked towards finding generators of the variety of $k-$NU posets for $k \geq 4$. Extending work of [3], they were able to find generating sets of the variety of $k-$NU posets with the so-called *strong selection property*. Let S_n be the poset we get from the the cube 2^n by removing its minimum element. Let S_n^{-1} be the poset we get from S_n by reversing its ordering. For posets P and Q, let $P \oplus Q$ be the ordinal sum of P and Q. It was shown in [20] that the variety of $k-$NU posets with the strong selection property is generated by the set consisting of S_n and S_n^{-1} for all $n \leq k-1$ and $S_n \oplus S_m^{-1}$ for all $n+m \leq k-1$. Recall that a poset is bounded if it has a minimum and a maximum element. It was further shown in [20] that the variety of bounded $k-$NU posets with the strong selection property is generated by the set consisting of $S_n \oplus S_m^{-1}$ for all $n+m \leq k-1$. Noting that there is a $7-$NU poset that does not have the strong selection property, the authors of [20] ask, for $n \geq 5$, for a generating set of the variety of bounded $k-$NU posets, and further ask if such a family could be finite.

In [22], clarity is brought to the situation when it is shown that a $k-$NU poset has the strong selection property if and only if 'its only zig-zags are polyads'. In the language of this paper, this translates to the statement that a $k-$NU poset $\mathbb{H}$ has the strong selection property if and only if the only critical H_∞-trees are *purely infinite* (i.e., all arcs are infinite), and are *polyads* (have only one uncoloured vertex).

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In [4], a poset P was defined to be *irreducible* if for any product $\prod_{i\in I} P_i$ of retracts P_i of P for which $P \trianglelefteq \prod_{i\in I} P_i$ there is some $i \in I$ with $P \trianglelefteq P_i$.

In [22], the relation between polyad zig-zags and the strong selection property is used to show that a bounded $k-$NU poset for $k = 5$ or 6 necessarily has the strong selection property. Referring to a characterisation from [17] of the finite irreducible posets having the strong selection property, a list of generators of the variety of bounded $k-$NU posets is given when $k = 5$ or 6. Again, this approach fails for $k \geq 7$, as $7-$NU posets need not have the strong selection property.

In this section, we show that in the case of an NU poset $\mathbb{H}$, the maximal critical H_∞-tree obstructions are purely infinite. We then show that the duals of such H_∞-trees are posets and give explicit descriptions of their cores. In the case that the alternating template of the purely infinite H_∞-tree is a polyad, the underlying digraph of the core is one of the posets $S_n \oplus S_m^{-1}$ of [20], thus reproving the result of [22] mentioned above, without the reference to [17] nor the strong selection property. Applying Theorem 8.1, we are then able to answer the question of [20] in the affirmative, and give a finite set of generators of the variety of bounded $k-$NU posets, for all k.

Recall that for an NU digraph $\mathbb{H}$, $\mathcal{M}_{\mathbb{H}}^{\infty}$ is the family of maximal elements of the family $\mathcal{T}_{\mathbb{H}}^{\infty}$ we get from the maximum duality $\mathcal{T}_{\mathbb{H}}$ for $\mathbb{H}$ by replacing ascending chains of trees with limiting H_∞-trees. An H_∞-tree ϕ is *purely infinite* if all of its arcs are infinite. The following holds for posets in particular.

Lemma 10.1. *Let $\mathbb{H}$ be a transitive $k-$NU digraph. Then the family $\mathcal{M}_{\mathbb{H}}^{\infty}$ consists of purely infinite H_∞-trees.*

Proof. By Lemma 7.1 it is enough to show for any critical obstruction $\mathbb{T}$ in $\mathcal{T}_{\mathbb{H}}$, that $\mathcal{T}_{\mathbb{H}}$ contains an $|H|$-witness of the purely infinite H_∞-tree over the same template.

Let $\mathbb{T}$ in $\mathcal{T}_{\mathbb{H}}$ have expansion ϕ over template $\mathbb{W}$, and assume a is an arc of $\mathbb{W} = \mathbb{W}(\phi)$ with $\phi(a) < |H|$. Consider the tree $\mathbb{T}'$ with expansion ϕ' over $\mathbb{W}$ where $\phi'(a) = |H|$ and $\phi'(a') = \phi(a')$ for all $a' \neq a$. We claim that $\mathbb{T}'$ is also a critical obstruction of $\mathbb{H}$, and so in $\mathcal{T}_{\mathbb{H}}$. Indeed, by the transitivity of $\mathbb{H}$ it is clear that $\mathbb{T} \not\to \mathbb{H}$ implies $\mathbb{T}' \not\to \mathbb{H}$. To prove $\mathbb{T}'$ is critical, notice that given any proper substructure of $\mathbb{T}'$, we can easily find a homomorphism $\mathbb{T}' \to \mathbb{T}$ to a proper substructure of $\mathbb{T}$. So $\mathbb{T}'$ is also in $\mathcal{T}_{\mathbb{H}}$.

It follows that the tree $\mathbb{T}^*$ with template $\mathbb{W}$ and expansion ϕ^* having $\phi^*(a) = |H|$ for all arcs a of $\mathbb{W}$ is also in $\mathcal{T}_{\mathbb{H}}$, as needed. □

In light of this result, the only H_∞-trees we consider in the rest of the section are purely infinite, and so we need only the alternating template $\mathbb{W}$ to represent them. The expansion ϕ is understood.

The following description of the dual of purely infinite H_∞-trees is an immediate transcription of Definition 6.2, only we have replaced the symbols 0 and ∞ with 1 and 0 respectively. As we will show that these duals are posets, this relabelling is in the interest of civility: if we did not do it, then we would have $(\infty, \ldots \infty) < (0, \ldots, 0)$.

Lemma 10.2. *Let $\mathbb{W}$ be a purely infinite H_∞-tree with arcset A. The dual digraph $\mathbb{K} = \mathbb{K}(\mathbb{W})$ is as follows.*

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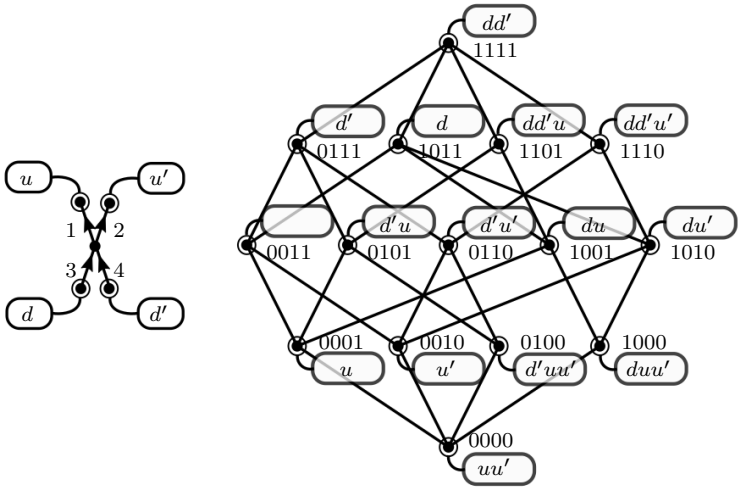

FIGURE 4. A purely infinite H_∞-tree $\mathbb{W}$ and its dual $\mathbb{K}(\mathbb{W})$.

Vertices: *The vertices of* $\mathbb{K}$ *are the* m*-tuples* x *of* $\prod_{a\in A}\{0,1\}$ *such that for each non-leaf* v *of* $\mathbb{W}$ *there is at least one out-arc* a *of* v *with* $x(a)=1$ *or one in-arc* a *of* v *with* $x(a)=0$.
Arcs: *A pair* (x,y) *of vertices of* $\mathbb{K}$ *is an arc if for all* $a\in\mathbb{W}$, $x(a)\le y(a)$.
Colours: *A vertex* x *gets the colour* h *if some leaf of* $\mathbb{W}$ *has colour* h, *and for every leaf* ℓ *of* $\mathbb{W}$ *having colour* h *we have, where* a *is the arc incident to* ℓ,

- a *is an out-arc of* ℓ *and* $x(a)=1$, *or*
- a *is an in-arc of* ℓ *and* $x(a)=0$.

□

Viewing the vertices of $\mathbb{K}$ as the elements of the poset $2^{|A|}$, where 2 is the two element chain $0\le\infty$, we see that (x,y) is an arc of $\mathbb{K}$ if and only if $x\le y$ in $2^{|A|}$. That is, $\mathbb{K}$ is the subposet of $2^{|A|}$ that it induces. This gives the following.

Lemma 10.3. *Let* $\mathbb{W}$ *be a purely infinite* H_∞*-tree. Then the underlying digraph of* $\mathbb{K}(\mathbb{W})$ *is a poset.*

The zig-zags defined in [21] are the transitive closure of our non-trivial purely infinite H_∞-trees. Since such an H_∞-tree maps to a transitive digraph $\mathbb{H}$ if and only if its transitive closure does, that these coincide is to be expected.

Observe that a transitive reflexive digraph is a poset if and only if it has no non-loop cycles, if and only if it has no two vertices with the same neighbourhood. Applying Lemmas 10.1 and 10.3 to Theorem 8.1 we thus get the following.

Corollary 10.4. *Let* D *and* $k\ge 3$ *be positive integers. Then there exists a finite set* $\mathcal{P}_{D,k}$ *of finite* $k-$NU *posets, (duals of non-trivial purely infinite* H_∞*-trees with at most* $k-1$ *leaves), such that any* $k-$NU *poset of diameter at most* D *is a retract of a finite product of digraphs in* $\mathcal{P}_{D,K}$. *Further, any* $k-$NU *transitive digraph of diameter at most* D *is a retract of a finite product of digraphs in* $\mathcal{P}_{D,K}\cup\{K_2\}$ *where* K_2 *is the symmetric reflexive clique on* 2 *vertices.* □

The following answers Problem 17 from [20] in the affirmative.

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Proposition 10.5. *Let $\mathbb{BNU}_k$ be the class of finite, bounded $k-$NU posets. Then there exist $\mathbb{P}_1,\dots,\mathbb{P}_n \in \mathbb{BNU}_k$ such that every poset in $\mathbb{BNU}_k$ is a retract of a finite product of the $\mathbb{P}_i$.*

Proof. By Theorem 8.1 it is enough to show that the set $\bigcup_{\mathbb{P}\in\mathbb{BNU}_k} \mathcal{M}^{\infty}_{\mathbb{P}}$ of purely infinite H_∞-trees with less than k leaves, that are critical for bounded $k-$NU posets, is finite; and that the duals of these H_∞-trees, which are posets by Lemma 10.3, are bounded.

The main point in showing both of these things is that a purely infinite H_∞-tree $\mathbb{W}$ that is critical for a bounded-above poset $\mathbb{P}$, cannot have an uncoloured sink v. Indeed, as $\mathbb{W}$ is critical, $\mathbb{W}\setminus\{v\}$ maps to $\mathbb{P}$; but then mapping v to the maximal element of $\mathbb{P}$ gives a homomorphism of $\mathbb{W}$ to $\mathbb{P}$. Similarly, $\mathbb{W}$ cannot have an uncoloured source.

As only leafs of $\mathbb{W}$ have colours, every non-leaf has an in-arc and an out-arc, and so the vertices $(0,0,\dots,0)$ and $(1,1,\dots,1)$ satisfy the vertex condition in Lemma 10.2. Thus $\mathbb{K}(\mathbb{W})$ is bounded.

Also, an alternating tree that has no sink nor source has no vertices of degree two, so having at most $k-1$ leaves, it can have at most $2k-4$ vertices, as we noted in Claim 7.5. So there are only finitely many such H_∞-trees. □

While the duals of our H_∞-trees are often cores, the duals of purely infinite H_∞-trees tend not to be. When one can, one would like to use the core of a dual – Theorem 8.1 would hold replacing the duals $\mathbb{K}(\phi)$ with their cores.

As mentioned above, a purely infinite H_∞-tree $\mathbb{W}$ is a *polyad* if it has only one non-leaf vertex. In the case that $\mathbb{W}$ is a purely infinite H_∞-polyad, we get a particularly nice description of $\mathbb{K}(\mathbb{W})$ and its core. We first observe the following.

Lemma 10.6. *For a $k-$NU poset $\mathbb{H}$, any purely infinite H_∞-polyad $\mathbb{W}$ in $\mathcal{M}^{\infty}_{\mathbb{H}}$ has distinct colours on its leaves.*

Proof. As $\mathbb{W}$ is a core we know that no two maximal leaves of $\mathbb{W}$, nor two minimal leaves can have the same colour; so we just have to show no maximal leaf and minimal leaf can have the same colour.

Assume that some minimal leaf d and some maximal leaf u do have the same colour h. For any other maximal leaf u', as $\mathbb{W}\setminus\{u\}$ maps to $\mathbb{H}$ with d mapping to h, the colour on u' is in the upset of h. Similarly the colour on any minimal leaf is in the downset of h. But then clearly there is a map of $\mathbb{W}$ to $\mathbb{H}$ (mapping the uncoloured vertex to h). □

For a set S let 2^S be the inclusion lattice of the family of subsets of S.

Proposition 10.7. *Let $\mathbb{W}$ be a purely infinite H_∞-polyad with distinct colours. Let U be the set of colours on its maximal leaves, and D the set of colours on its minimal leaves. The dual $\mathbb{K}=\mathbb{K}(\mathbb{W})$ is (isomorphic to)*

$$2^D \times ((2^U)^{-1}) \setminus \{(D,U)\}.$$

The core of $\mathbb{K}$ is (isomorphic to)

$$(2^D \setminus \{D\}) \oplus (2^U \setminus \{U\})^{-1}.$$

(See Figures 4 and 5 for an example.)

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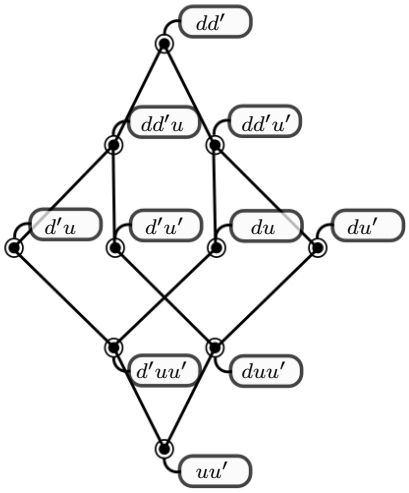

FIGURE 5. The core of the dual $\mathbb{K}(\mathbb{W})$ of the purely infinite H_∞-polyad shown in Figure 4.

Proof. Let v be the non-leaf vertex of $\mathbb{W}$. As the colours are distinct and each arc has one, we refer to an arc by the colour on its leaf. So colours in U refer to out-arcs of v and colours in D refer to in-arcs of v.

As out-arcs of v are in-arcs of their leaf, Lemma 10.2 tells us that a vertex x of $\mathbb{K}$ has the colour $u \in U$ if and only if $x(u) = 0$, and has the colour $d \in D$ if and only $x(d) = 1$. So any vertex of $\mathbb{K}$ is uniquely defined by the set of colours in $2^D \times 2^U$ that it supports; further, the ordering of $\mathbb{K}$ is that induced from $2^D \times (2^U)^{-1}$. Thus our first statement is immediate by the simple observation that the only vertex violating the vertex condition in the definition of $\mathbb{K}$ is the vertex with all colours.

To show the second statement we will show that a vertex of $\mathbb{K}$ is in the core if and only if it has all colours U or all colours D. The subset of $\mathbb{K}$ induced by vertices with all colours in U is $2^D \setminus \{D\}$ and that induced by vertices with all colours in D is $(2^U)^{-1} \setminus \{U\}$, so this is enough.

Observe first, that the maximum element of $\mathbb{K}$ is the vertex coloured with all and only colours in D, and any vertex with colours only in D retracts up to it. Similarly any vertex with colours only in U retracts down to the minimum element, which is coloured by all and only colours is U.

Next we show that a vertex that has all colours U or all colours D, except clearly the vertex with colours $U \cup D$, is in the core of $\mathbb{K}$. Towards contradiction, let x have colours U and a proper subset D_x of the colours D, and assume that it retracts to some other vertex x' under a retraction of $\mathbb{K}$ to its core. Further, assume that x is a maximal such counter-example. The vertex x' must have all colours in $U \cup D_x$ and some other colour $d' \in D \setminus D_x$. As x' is not the vertex with colours $U \cup D$, (as this is not in $\mathbb{K}$), there is some other colour $d \in D \setminus D_{x'}$. But then x is below the vertex z with colours $U \cup D_x \cup \{d\}$, while x' is not. By the maximality of x, z is in the core, and so x cannot retract to x'.

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Finally, we show that a vertex that does not have all colours U or all colours D, is not in the core. Towards contradiction, assume that x is a maximal such element in the core; x has some proper subset U_x of the colours U and some proper subset D_x of the colours D. As x is in the core, it does not retract to a cover, so must have at least two covers: a and b. Being above x they have a superset of its colours in D and a subset of its colours U_x. As x was a maximal counter-example, they both have all colours D, and incomparable subsets U_a and U_b of the colours U_x.

But then consider the vertex $a \wedge b$ in $\mathbb{K}$ with colours $D \cup (U_a \cap U_b)$. By choice of x, it is in the core; and it is between x and a, contradicting the fact that a covers x. □

11. Concluding Remarks

We recently learned that finite duals for the family of trees not mapping to a given structure $\mathbb{H}$ were constructed in [5]. Their construction, being much more general, is less explicit than ours. In the case of the tree duality for the directed arc $\mathbb{A}$, which was given as an example in Section 5, we have verified that the dual constructed in [5] is the same as ours. It would be interesting to see if this is true for all $k-$NU reflexive digraphs $\mathbb{H}$.

In general the duals of [5] and our duals must be homomorphism equivalent, so have the same core, but they need not be the same. As such, it would also be interesting to extend Proposition 10.7 and get an explicit description of the core of $\mathbb{K}(\phi)$ for any H_∞-tree ϕ. In results such as Theorem 8.1 and Corollary 8.2 the duals $\mathbb{K}(\phi)$ can be replaced with their cores- the cores also generate the variety of reflexive NU-digraphs.

In fact, in the case of posets it follows from [4], as is observed in [22], that the variety of NU-posets is generated by the set of irreducible NU-posets. The theme of [4] is that irreducibles are a good, smallest, generator of a variety. Once one finds the core of $\mathbb{K}(\phi)$ for a more general H_∞-tree ϕ, it would be interesting to decide if it is irreducible.

Strengthening the definition of irreducible, a reflexive digraph is *completely irreducible* if for any product $\prod_{i\in I}\mathbb{H}_i$ for which $\mathbb{H} \trianglelefteq \prod_{i\in I}\mathbb{H}_i$ there is some $i \in I$ with $\mathbb{H} \trianglelefteq \mathbb{H}_i$. (The $\mathbb{H}_i$ need not be retracts of $\mathbb{H}$.) A reflexive digraph is *strongly irreducible* if the same holds, but only for finite products. These concepts were considered in [10]. Very clearly, the completely irreducible digraphs in a variety must generate it.

By Theorem 8.1, the variety of $k-$NU reflexive digraphs is generated by strongly irreducible elements, and any strongly irreducible $k-$NU digraph is a retract (after removing colours) of the (core of the) dual of an H_∞-tree with at most $k-1$ leaves.

In an upcoming paper we consider such problems as describing the strongly irreducible $k-$NU digraphs, showing they are completely irreducible, and comparing them to irreducible $k-$NU digraphs.

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