Note on Robust Critical Graphs with Large Odd Girth

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Abstract

A graph G is $(k+1)$-critical if it is not k-colourable but $G-e$ is k-colourable for any edge $e \in E(G)$. In this paper we show that for any integers $k \geq 3$ and $\ell \geq 5$ there exists a constant $c = c(k, \ell) > 0$, such that for all $\tilde{n}$, there exists a $(k+1)$-critical graph G on n vertices with $n > \tilde{n}$ and odd girth at least ℓ, which can be made $(k-1)$-colourable only by the omission of at least cn^2 edges.

Key words: Graph, colour critical, odd girth.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A *k-colouring* χ of G is a function $\chi : V(G) \to [k]$ such that $\chi(u) \neq \chi(v)$ whenever $\{u, v\}$ is an edge of G. The graph G is called *k-colourable* if there exists a k-colouring of G, and is called *k-chromatic* if k is the smallest integer such that G has a k-colouring. G is called *$(k+1)$-critical* if it is not k-colourable, but $G - e$ is k-colourable for any edge e of G. The *girth* of G, denoted by $g(G)$, is the length of the shortest cycle in G.

In 1970, Toft [9] extended a result of Dirac [2] to show that for $k \geq 3$, there exists some constant c_k such that for any n, there exists a $(k+1)$-critical graph on n vertices with at least $c_k n^2$ edges. These graphs constructed by Toft and

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Dirac had large bipartite subgraphs and contained small odd cycles. Erdős [4] suggested that this may always be the case.

For $k = 3$, the above mentioned 4-critical graphs produced by Toft, could be made bipartite by removing cn edges, for some constant $c > 0$. This led Toft to ask whether there exists some constant $d > 0$ such that for n large, there exists a 4-critical graph on n vertices from which one must remove dn^2 edges in order to make it 2-colourable. This question was answered by Stiebitz and Rödl.

In 1987, Stiebitz [8], showed the existence of $(k+1)$-critical graphs, for $k \geq 3$, from which one must remove at least $d_k n^2$ edges to make them $(k-1)$-colourable. The graphs that Stiebitz constructed contained many triangles.

At about the same time, Rödl [7] (unpublished) showed that for any $\ell \geq 3$ there are 4-critical graphs with odd girth at least ℓ, from which one must remove at least $d_k n^2$ edges in order to make them 2-colourable.

In this paper, we extend Rödl's unpublished construction and prove the following theorem:

Theorem 1 *For any integers $k \geq 3$ and $\ell \geq 5$ there exists a constant $c = c(k, \ell) > 0$, such that for all $\tilde{n}$, there exists a $(k+1)$-critical graph G on n vertices with $n > \tilde{n}$ and odd girth at least ℓ, which can be made $(k-1)$-colourable only by the omission of at least cn^2 edges.*

Recall that by a result of Kövári, Sós, and Turán [5], any graph with more than $\frac{1}{2}(n^{\frac{3}{2}} + n - n^{\frac{1}{2}})$ edges has a 4-cycle, so we cannot hope to replace 'odd girth' in the above theorem with 'girth'.

Unfortunately, the proof of Theorem 1 gives only a weak lower bound on the constant $c(k, \ell)$, and we make no effort to evaluate it. In section 5 we give an easy upper bound on $c(3, \ell)$ as consequence of a result of Andrásfai, Erdős, and Sós [1].

2 Preliminaries

Before we proceed with the proof of Theorem 1 we give two preliminary lemmas that will be used later in the proof. Note that when χ is a map with domain V, and $U \subset V$, we will write $\chi(U)$ for the set $\{\chi(u) \mid u \in U\}$.

A set $\mathcal{A}$ of mappings from W to $[k]$ is said to be *closed under permutations of* $[k]$ if for every $\alpha \in \mathcal{A}$ and any permutation σ of $[k]$, $\sigma \circ \alpha \in \mathcal{A}$.

We start with Lemma 2 which is a slight variation of a result of Müller [6]. Müller proved that under the assumptions of Lemma 2, conditions (i) and (ii) hold.

Lemma 2 *Let $k, \ell \in \mathbb{N}$, $k \geq 3$, and let $\mathcal{A} \subseteq \{\alpha \mid \alpha : W \to [k]\}$ be closed under permutation of $[k]$. Then, there exists a k-chromatic graph $M = M(W, \mathcal{A}, k, \ell)$, with $W \subset V(M)$, which satisfies the following properties.*

(i) $g(M) \geq \ell$
(ii) $\{\chi|_W : \chi$ *is a* k*-colouring of* $M\} = \mathcal{A}$
(iii) The distance between any two vertices of W *is at least* ℓ*.*

PROOF.

Given $k, \ell, \mathcal{A}$, and W, as in the statement of the lemma, the result in [6] guarantees the existence of a k-chromatic graph M_0, with $W \subset V(M_0)$, which satisfies properties (i) and (ii). To prove the lemma, we will construct a new graph M from M_0, which satisfies properties (i) and (ii), and also satisfies property (iii). This construction is a standard technique.

Before we construct M, we will define an auxiliary graph R. First, let R' be a $(k+1)$-critical graph with $g(R') \geq \ell$; such graphs exist by [3]. Let $e' = \{x, x'\}$ be an edge of R'. Then $R' - e'$ is a k-chromatic graph such that for any k-colouring χ of $R' - e'$, $\chi(x) = \chi(x')$. Now let $R = (R' - e') \cup \{x', y\}$, where y is a new vertex adjacent to x'. Then the graph R has $g(R) \geq \ell$, is k-chromatic, and has distance $d_R(x, y) \geq \ell$. Moreover, for any k-colouring χ of R, $\chi(x) = \chi(x') \neq \chi(y)$.

We now construct the graph M from M_0 and several copies of R.

Construction 3 *Let E_W be the set of edges of M_0 with at least one vertex in W. For each $e = \{u, w\} \in E_W$, let R_e be a copy of R, and let x_e and y_e denote the copies of x and y in R_e. Replace the edge e in M_0 with the graph R_e, by identifying u with x_e and w with y_e. Let the resulting graph be called M.*

Note that by Construction 3 all of the vertices of M_0 are vertices of M, and so in particular $W \subset V(M_0) \subset V(M)$. We now verify that M satisfies properties (i) - (iii) of the lemma.

(i): Recall that the graph M_0 and all the copies of R have girth at least ℓ. Therefore, any cycle in M which is not entirely in M_0 or a copy of R, must contain the vertices x_e and y_e of R_e for some $e \in E_W$. Since $d(x_e, y_e) \geq \ell$ in R_e and hence in M, it follows that $g(M) \geq \ell$.

(ii): We first show that any k-colouring of M_0, can be extended to a k-colouring of M. Then conversely, we show that any k-colouring of M induces a k-colouring of M_0. This will then give property (ii) since by [6], $\{\chi_0|_W : \chi_0 \text{ is a } k\text{-colouring of } M_0\} = \mathcal{A}$.

Let χ be a k-colouring of M_0. For any edge $\{a, b\} \in E(M_0)$, we clearly have that $\chi(a) \neq \chi(b)$. Since the only edges of M that are not in M_0 are those which are in copies of R, it remains to show that for any $e = \{u, w\} \in E_W$, χ can be extended to a proper k-colouring of R_e. Since χ is a k-colouring of M_0, and e is in M_0, $\chi(x_e) = \chi(u) \neq \chi(w) = \chi(y_e)$. Since $\chi(x_e) \neq \chi(y_e)$, χ can be extended to a k-colouring of R_e.

On the other hand, let χ be a k-colouring of M. The only edges of M_0 that are not in M are those of E_W, and for any $e = \{u, w\} \in E_W$ we have that $\chi(u) = \chi(x_e) \neq \chi(y_e) = \chi(w)$. Thus χ induces a k-colouring of M_0.

(iii): Every edge e, incident to a vertex of W, was replaced in Construction 3 with a copy R_e of R, by identifying the endpoints of e with the vertices x_e and y_e of R_e. Since $d(x_e, y_e) \geq \ell$ in R_e, any two vertices of W in the graph M are also at a distance of at least ℓ.

This completes the proof of the lemma. □

Lemma 2 gives no relation between the size of W and the number of vertices of M. In the proof of Theorem 1 we will need Lemma 4, which is a modification of Lemma 2. In Lemma 4 the number of vertices of the graph will be bounded by a constant (depending only on k and ℓ) times the size of the input set $U \cup \{u^*\}$ (which replaces the set W). It will be sufficient to prove such a strengthening only for sets $\mathcal{A}$ of a special form, as specified by condition (ii) below.

Lemma 4 *Let $d, k, \ell \in \mathbb{N}$ be such that $d \geq 1$, $k \geq 3$, $\ell \geq 5$, and ℓ is odd. Then there exists a graph $T = T(d, k, \ell)$, with a distinguished subset of vertices $U \cup \{u^*\} \subset V(T)$, where $u^* \notin U$ and $|U| = k^d$, and a constant $m(k, \ell)$, such that the following properties hold.*

(i) $g(T) \geq \ell$
(ii) *Any mapping $\chi : U \cup \{u^*\} \to [k]$ can be extended to a k-colouring of T if, and only if, $\chi(u^*) \in \chi(U)$.*
(iii) *The distance between any two vertices of $U \cup \{u^*\}$ is at least ℓ.*
(iv) $|V(T)| < k^d \cdot m(k, \ell)$

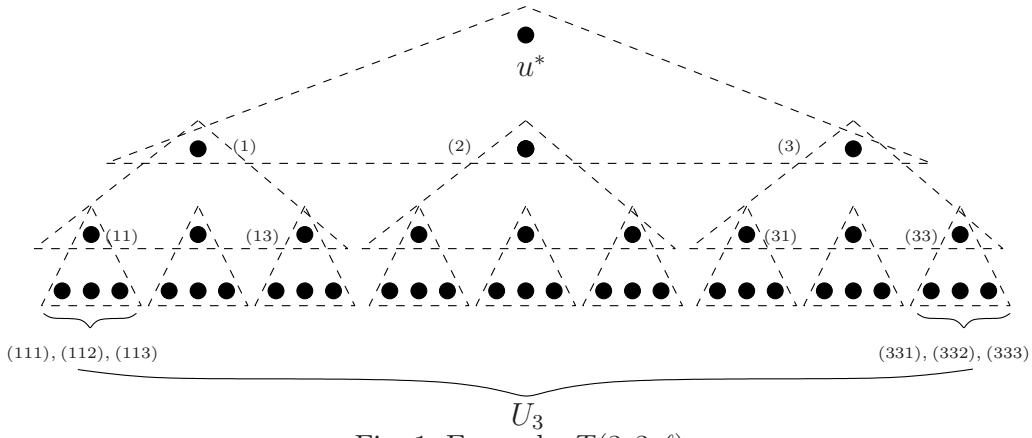

Fig. 1. Example: $T(3, 3, \ell)$.

PROOF.

Let k and ℓ be given. Let $V = \{v_1, \ldots, v_k\}$ be a set of independent vertices and let $v \notin V$. Set $W = V \cup \{v\}$ and define the set of mappings

$$\mathcal{A} = \{\alpha : W \to [k] \mid \alpha(v) \in \alpha(V)\}.$$

We now apply Lemma 2 and obtain the graph $M = M(W, \mathcal{A}, k, \ell)$. The graph M thus contains all the vertices of $V \cup \{v\}$ and has the property that $\chi : V \cup \{v\} \to [k]$ can be extended to a proper k-colouring of M if, and only if, $\chi(v) \in \chi(V)$.

We will construct the graph $T = T(d, k, \ell)$ by taking several copies of the graph M and gluing them together as described in Construction 5 below. The copies of M will be pairwise vertex disjoint except for those vertices identified in Construction 5.

Setup for Construction 5 For all $i = 1, \ldots, d$, let $U^i = \Big\{(\xi_1 \ldots \xi_i) \;\mid\; 1 \leq \xi_j \leq k, \text{ for } j = 1, \ldots, i \Big\}$. Furthermore, let $U = U^d$. We will view U^i as a set of independent vertices, with which we will identify some of the vertices of T.

Construction 5 *For all $i = 1, \ldots, d-1$ and $1 \leq \xi_j \leq k$, where $j = 1, \ldots, i$, let $M_{(\xi_1 \ldots \xi_i)}$ be a copy of M. For each $M_{(\xi_1 \ldots \xi_i)}$, identify the copy of V with $\{(\xi_1 \ldots \xi_i\ 1), \ldots, (\xi_1 \ldots \xi_i\ k)\}$ and the copy of v with $(\xi_1 \ldots \xi_i)$. Moreover, let M_{u^*} be a copy of M where we identify v with u^* and V with U^1.*

Observe that with these identifications any two copies of M share at most one vertex (see Figure 1). We verify now that T satisfies properties (i) - (iv) required by the lemma.

(i): Let C be some cycle in T. Since every edge of T is in a copy of M, either

C is completely within some copy of M, or C contains two vertices of the set $W = V \cup \{v\}$ in some copy of M. In the first case C has girth at least ℓ, because $g(M) \geq \ell$ by Lemma 2 (i). In the second case, C has girth at least C because the distance between any two vertices of W is at least ℓ by Lemma 2 (iii). This was for any cycle C in T, so $g(T) \geq \ell$.

(ii): Let $(1 \dots 1)_i$ denote a vector of i ones. By the definition of $\mathcal{A}$ from the beginning of the proof, any mapping $\chi : U \cup \{u^*\} \to [k]$ can be extended to a k-colouring of $M_{(\xi_1 \dots \xi_i)}$ for all $i = 1, \dots, d-1$, if and only if

$$\chi((\xi_1 \dots \xi_i)) \in \Big\{\chi(\xi_1 \dots \xi_i\ 1), \dots, \chi(\xi_1 \dots \xi_i\ k)\Big\}. \tag{a}$$

Similarly, χ can be extended to a k-colouring of M_{u^*} if, and only if,

$$\chi(u^*) \in \{\chi(1), \dots, \chi(k)\}. \tag{b}$$

To prove the 'only if' implication of (ii), we consider $\chi : U \cup \{u^*\} \to [k]$, such that $\chi(u^*) \in \chi(U)$. Assume w.l.o.g. that $\chi(u^*) = 1 = \chi((1 \dots 1)_d)$, where $(1 \dots 1)_d \in U$. We must show that χ can be extended to a k-colouring of T.

Proceeding backwards for $i = d-1, \dots, 1$ we do as follows. We define χ on the set U^i by setting $\chi((\xi_1, \dots, \xi_i)) = \chi((\xi_1, \dots, \xi_i, 1))$ for for all $(\xi_1 \dots \xi_i) \in U^i$. Since this makes (a) true, we can extend χ to a k-colouring of $M_{(\xi_1 \dots \xi_i)}$ for all $(\xi_1 \dots \xi_i) \in U^i$. Furthermore, we get that $\chi((1, \dots, 1)_i) = \chi((1, \dots, 1)_{i+1}) = 1$. Having done this for $i = 1$, we have $\chi(1) = 1$. Since by assumption $\chi(u^*) = 1$, this makes (b) true, and so we can extend χ to a k-colouring of M_{u^*}. We have thus extended χ to a k-colouring of all of T.

To prove the 'if' implication of (ii), let $\chi : U \cup \{u^*\} \to [k]$ be such that $\chi(u^*) = 1 \notin \chi(U)$. Towards a contradiction, assume that χ can be extended to a k-colouring of T. Since there is a vertex of colour 1 in $\{u^*\}$ and there are none in $U^d = U$, there is a maximal i with $i < d$ such that $1 \in \chi(U^{i-1})$ and $1 \notin \chi(U^i)$. Without loss of generality let $\chi((1 \dots 1)_{i-1}) = 1$. Thus, $M_{(1 \dots 1)_{i-1}}$ is k-coloured with $\chi((1 \dots 1)_{i-1}) = 1$ while,

$$1 \notin \{\chi((1 \dots 1\ 1)_i), \dots, \chi((1 \dots 1\ k)_i)\}.$$

This, however, contradicts property (a).

(iii): Since every edge of T is in a copy of M, the shortest path between two vertices of $U \cup \{u^*\}$ is either completely within some copy of M, or passes through two vertices of $W = V \cup \{v\}$ in some copy of M. Again by Lemma 2

(iii), the length of such a path is at least ℓ.

(iv): Set $m(k,\ell) = |V(M)|/(k-1)$. Since every vertex of T is in at least one of the $(k^d-1)/(k-1)$ copies of M, we have that $|V(T)| \leq (\frac{k^d-1}{k-1}) \cdot |V(M)| < k^d \cdot m(k,\ell)$.

Therefore, properties (i) - (iv) hold, and thus, the proof of the lemma is complete. □

3 The Graph $\hat{G} = \hat{G}(d,k,\ell)$

In this section, we construct a graph $\hat{G} = \hat{G}(d,k,\ell)$, which depends on the integers d, k, and ℓ. In Lemma 7, we observe some important properties of $\hat{G}$.

Given k and ℓ, any $(k+1)$-critical subgraph G of $\hat{G}(d,k,\ell)$, for an appropriate choice of d, will then satisfy the properties of Theorem 1. This will be proved in Section 4.

Graph $\hat{G}$ will be constructed from several components. We now define these components before giving the actual construction.

Setup for Construction 6 Let the integers $k \geq 3$, $\ell \geq 3$ and d be fixed.

(i) Erdős [3] showed that there exist k-critical graphs with girth at least ℓ. Among such graphs let $F = F(k,\ell)$ be one with the fewest vertices (thus F is connected). Set $f(k,\ell) = |V(F)|$ and $V(F) = \{v_1, \ldots, v_{f(k,\ell)}\}$.

(ii) Let F_B be the blowup of F, that is, the graph defined by replacing each vertex $v_i \in V(F)$ with a set B_i, of k^d independent vertices, and each edge $\{v_i, v_j\} \in E(F)$ with the complete bipartite graph between the sets B_i and B_j. In other words, the graph F_B has vertex set $V(F_B) = \bigcup_{i=1}^{f(k,\ell)} B_i$ and edge set

$$E(F_B) = \Big\{ \{b_i, b_j\} \mid b_i \in B_i, b_j \in B_j, \{v_i, v_j\} \in E(F) \Big\}.$$

(iii) For all $i = 1, \ldots, f(k,\ell)$, let T_i be a copy of the graph $T(d,k,\ell)$ provided by Lemma 4. Let U_i and u_i^* be the copies of U and u^* respectively, in T_i.

(iv) Set $W = \{w_1, \ldots, w_{f(k,\ell)}\}$ and let $\mathcal{A}$ be the set of all mappings $\alpha : W \to [k]$, that are 'inconsistent with respect to F'; that is, $\alpha \in \mathcal{A}$ if, and only if, the mapping $\chi : V(F) \to [k]$ defined by $\chi(v_i) = \alpha(w_i)$ for all $i = 1, \ldots, f(k,\ell)$, is not a proper k-colouring of F. Let $H = M(W, \mathcal{A}, k, \ell)$

be the k-chromatic graph returned by Lemma 2 for this choice of $\mathcal{A}, k, \ell$, and W, and set $h(k,\ell) = |V(H)|$.

We now give the construction.

Construction 6 *Given integers d, k, and ℓ, let $F = F(k,\ell)$ and $f(k,\ell)$ be defined as in item (i) of Setup for Construction 6.*

Construct $\hat{G} = \hat{G}(d,k,\ell)$ from pairwise disjoint components F_B, $T_1, \ldots, T_{f(k,\ell)}$, and H, defined in items (ii), (iii) and (iv) of Setup for Construction 6, by making the following identifications.

(i) For $i = 1, \ldots, f(k,\ell)$, identify U_i of T_i with B_i of F_B.
(ii) For $i = 1, \ldots, f(k,\ell)$, and $j = 1, \ldots, k^d$, identify u_i^* of T_i with w_i of H.

Thus, the graph $\hat{G}$ has vertex set

$$V(\hat{G}) = \bigcup_{i=1}^{f(k,\ell)} V(T_i) \cup (V(H) - W)$$

and edge set

$$E(\hat{G}) = (\bigcup_{i=1}^{f(k,\ell)} E(T_i)) \cup E(F_B) \cup E(H).$$

Set $C(k,\ell) = f(k,\ell) \cdot m(k,\ell) + h(k,\ell)$ where $m(k,\ell)$ is the constant from Lemma 4, which is independent of d. In the following lemma, we observe some important properties of Construction 6.

Lemma 7 *Let the integers k and ℓ be given. Let F and H, be as in Setup for Construction 6. Then for any positive integer d, the graph $\hat{G} = \hat{G}(d,k,\ell)$, of Construction 6, has the following properties:*

(i) $\hat{G}$ is not k-colourable.
(ii) For any edge $e \in F_B$, $\hat{G} - e$ is k-colourable.
(iii) $\hat{G}$ has odd girth at least ℓ.
(iv) $|V(\hat{G})| < C(k,\ell) \cdot k^d$.
(v) $\hat{G}$ cannot be made $(k-1)$-colourable without removing at least k^{2d} edges of F_B.

PROOF. We will now show that for any positive integer d, the graph $\hat{G} = \hat{G}(d,k,\ell)$ satisfies properties (i) - (v).

(i) *$\hat{G}$ is not k-colourable.*

Towards a contradiction, assume that $\hat{G}$ has a k-colouring χ. For $i = 1, \ldots, f(k, \ell)$, χ induces a k-colouring of T_i, so by property (ii) of Lemma 4, $\chi(u_i^*) \in \chi(U_i)$. By the identification of u_i^* with w_i, and U_i with B_i, there is thus some $b_i \in B_i$ such that $\chi(w_i) = \chi(b_i)$.

The set of vertices $\{b_i \mid i = 1, \ldots, f(k, \ell)\}$ induces a copy F_1, of F, in $\hat{G}$, and χ induces a k-colouring of F_1. Thus, by the definition of $\mathcal{A}$ in Setup for Construction 6 (iv), χ restricted to $W = \{w_1, \ldots, w_{f(k,\ell)}\}$ is not in $\mathcal{A}$. This contradicts the fact that χ induces a k-colouring of H. Therefore, $\hat{G}$ is not k-colourable.

(ii) *For any edge $e \in F_B$, $\hat{G} - e$ is k-colourable.*

Without loss of generality assume that $e = \{b_1, b_2\}$, where b_1 is in B_1 and b_2 is in B_2. Recall that F_B is the blowup of F, so B_1 and B_2 correspond to vertices v_1 and v_2 in F.

Since $g(F) = \ell \geq 5$ there is some vertex v_i of F that is distance at least two from both vertices v_1 and v_2. Assume, w.l.o.g., that $i = f(k, \ell)$. Thus, there are no edges from $B_{f(k,\ell)}$ to B_1 or B_2. Since F is k-critical, there exists a k-colouring χ' of the graph F in which the only vertex that gets the colour k is $v_{f(k,\ell)}$.

Consequently, setting

(a) $\chi(b) = \chi'(v_i)$ where $b \in B_i$, for any $b \in V(F_B) - \{b_1, b_2\}$, and
(b) $\chi(b_1) = \chi(b_2) = k$,

defines a k-colouring χ of $F_B - e$. Since U_i is identified with the vertex set B_i of F_B, this defines χ on U_i for all $i = 1, \ldots, f(k, \ell)$.

We then extend χ to the rest of $\hat{G} - e$ as follows.

(e) For $i = 1, 2$, since $k \in \chi(U_i)$, we can extend χ to a k-colouring of T_i in which $\chi(u_i^*) = k$.
(f) For $i \neq 1, 2$, we extend χ to any k-colouring of T_i.
(g) The vertices w_1 and w_2, by their identification with u_1^* and u_2^*, both get the colour k under χ. Since $\{v_1, v_2\}$ is an edge in F, χ then induces on $\{w_1, \ldots, w_{f(k,\ell)}\}$ a mapping that is inconsistent with F. Thus, by the choice of $\mathcal{A}$ in Construction 6 (iii), χ can be extended to a k-colouring of H.

This exhibits a k-colouring of $\hat{G} - e$, so $\hat{G} - e$ is k-colourable.

(iii) $\hat{G}$ *has odd girth at least* ℓ.

First note that the components of $\hat{G}$: the blowup F_B, the graphs T_i, for all $i = 1, \ldots, f(k,\ell)$, and the graph H, have odd girth at least ℓ. Now let C be some odd cycle of $\hat{G}$ that is not completely contained within one of the above components. The cycle C must contain a vertex v in $V(T_i) - (V(B) \cup V(H))$ for some i. Since the cycle C is not entirely within the graph T_i, v must be contained on a path of C containing at least two vertices of $U_i \cup \{u_i^*\}$. By property (iii) of Lemma 4 these vertices are at least a distance ℓ apart. Thus C must have length at least ℓ.

(iv) $|V(\hat{G})| < C(k,\ell) \cdot k^d$.

Since every vertex of F_B was identified with a vertex of T_i, for some $i \in \{1, \ldots, f(k,\ell)\}$, and $\hat{G}$ was constructed from F_B, H, and $f(k,\ell)$ copies of T each of which had $|V(T)| < k^d \cdot m(k,\ell)$ vertices by Lemma 4 (iv), it follows that $|V(\hat{G})| < f(k,\ell) \cdot k^d \cdot m(k,\ell) + h(k,\ell) < C(k,\ell) \cdot k^d$.

(v) $\hat{G}$ *cannot be made* $(k-1)$*-colourable without removing at least* k^{2d} *edges of* F_B.

To allow the graph $\hat{G}$ to be $(k-1)$-colourable, we need to remove at least one edge from each copy of F. Restricting our attention to copies of F in $F_B \subset \hat{G}$ with at most one vertex in each set B_i, we find that there are at least $(k^d)^{f(k,\ell)}$ copies of F. As any edge in F_B is in at most $(k^d)^{f(k,\ell)-2}$ of these copies of F, we need to remove at least k^{2d} edges to be sure we have removed at least one from each copy of F. $\square$

4 Proof of Theorem 1

We now proceed with the proof of the main theorem.

PROOF. Given $k \geq 3$ and $\ell \geq 5$, let $f(k,\ell)$ and F_B be as in Setup for Construction 6 (i) and (ii), and let $C(k,\ell)$ be the constant from Lemma 7. Set $c(k,\ell) = (\frac{1}{C(k,\ell)})^2$. For $\tilde{n}$ given, let d be any integer such that

$$\tilde{n} \leq f(k,\ell) \cdot k^d.$$

Let G be a $(k+1)$-critical subgraph of the graph $\hat{G} = \hat{G}(d,k,\ell)$ provided by Construction 6, and let $n = |V(G)|$. Note that by property (ii) of Lemma 7,

G has all the vertices and the edges of F_B. Therefore,

$$f(k,\ell)\cdot k^d \leq n,$$

and by property (v) of Lemma 7, we must remove at least k^{2d} edges to make G $(k-1)$-colourable. Since property (iii) of Lemma 7 gives that the odd girth of $\hat{G}$ is at least ℓ, it follows that the same holds for the odd girth of G. Furthermore,

$$n \leq |V(\hat{G})| < C(k,\ell)\cdot k^d$$

by property (iv) of Lemma 7, and thus the choice of $c(k,\ell)$ yields that

$$c(k,\ell)\cdot n^2 = (\frac{1}{C(k,\ell)})^2 \cdot n^2 < k^{2d}.$$

Therefore, we must remove at least $c(k,\ell)\cdot n^2$ edges from G to make it $(k-1)$-colourable. We have now verified all the necessary properties and thus the proof is complete. $\Box$

5 Concluding Remarks

The construction of the graph $\hat{G}$ together with Theorem 1 gives a lower bound for the constant c. It may be of interest to obtain a better estimate. A straightforward upper bound on $c(3,\ell)$ follows from a result of Andrásfai, Erdős, and Sós [1] who proved that every nonbipartite graph G on n vertices with odd girth at least ℓ, must have minimum degree $\delta \leq \frac{2}{\ell}\cdot n$. Indeed, sequentially deleting vertices of minimum degree as long as each remaining graph is nonbipartite, we delete at most

$$2(\frac{n}{\ell} + \frac{n-1}{\ell} + \cdots + \frac{1}{\ell}) \approx \frac{n^2}{\ell}$$

edges. Thus, $c(3,\ell) \leq (1+o(1))/\ell$, where $o(1)$ is a function of n that goes to 0 as n goes to infinity.

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