# ON RAMSEY MINIMAL GRAPHS

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Abstract. A graph $G$ is $r$-ramsey-minimal with respect to $K_k$ if every $r$-colouring of the edges of $G$ yields a monochromatic copy of $K_k$, but the same is not true for any proper subgraph of $G$. In this paper we show that for any integer $k \geq 3$ and $r \geq 2$, there exists a constant $c > 1$ such that for large enough $n$, there exist at least $c^{n^2}$ non-isomorphic graphs on at most $n$ vertices, each of which is $r$-ramsey-minimal with respect to the complete graph $K_k$. Furthermore, in the case $r = 2$, we give an asymmetric version of the above result.

## 1. Introduction

A classical problem of graph theory is determining the ramsey number for a graph $H$. This is the minimum number $n$ such that any 2-colouring the edges of the complete graph $K_n$ on $n$ vertices yields a monochromatic copy of $H$.

A variation on this problem is to look at colouring the edges of some graph $G$ instead of $K_n$. Again we are interested in the minimal graphs that yield a monochromatic copy of $H$ when two coloured.

A graph $G$ is *ramsey with respect to* $H$ if any two colouring of its edges yields a monochromatic copy of $H$. It is *ramsey-minimal with respect to* $H$ if it is ramsey, but none of its proper subgraphs are. For results about ramsey-minimal graphs, see, for example, [1] - [10], [12], [13], [15], [16], and [19].

In [7], Burr, Erdős, and Lovász showed that for any integer $k \geq 3$ there are infinitely many non-isomorphic graphs that are ramsey-minimal with respect to the complete graph $K_k$ on $k$ vertices. In [10], this result was strengthened to show that for any integer $k \geq 3$ there exists a constant $c > 1$ such that for all $n$ large enough, there exist at least $c^{nlogn}$ non-isomorphic graphs on at most $n$ vertices that are ramsey-minimal with respect to $K_k$.

A graph $G$ is *$r$-ramsey with respect to $H$* if any $r$-colouring of its edges yields a monochromatic copy of $H$. It is *$r$-ramsey minimal with respect to $H$* if it is $r$-ramsey, but none of its proper subgraphs are.

In this paper we strengthen the result of [10] stated above, and extend it to an arbitrary number of colours. Our main result is as follows.

**Theorem 1.1.** *For any integers $r \geq 2$ and $k \geq 3$ there exist constants $c = c(r,k) > 1$ and $n_0(r,k)$ such that for all $n \geq n_0(r,k)$, there exist at least $c^{n^2}$ non-isomorphic graphs on at most $n$ vertices, that are $r$-ramsey-minimal with respect to $K_k$.*

A common generalisation for ramsey-minimal graphs is to the following non-symmetric definition. A graph $G$ is *ramsey with respect to $H_1$ and $H_2$* if any 2-colouring yields a monochromatic copy of $H_1$ in the first colour, or a monochromatic copy of $H_2$ in the second colour.

At the end of the paper, in Section 6, we prove a non-symmetric version of the result for the case $r = 2$. That is, we observe that with similar quantifications as in Theorem 1.1 there are many graphs that are ramsey-minimal with respect to $K_\ell$ and $K_k$.

Indeed, the results of [7] and [10] above were also proved in this more general setting.

The simplest case of Theorem 1.1 is $k = 3$ and $r = 2$. We prove this case in Section 3. In Section 4 we prove the theorem for $r = 2$ and all $k$. Finally in Section 5 we complete the proof. We begin by introducing a preliminary result, the proof of which we put off to Section 7.

## 2. Preliminaries

Throughout the paper, we will identify a graph or hypergraph $G$ with its edgeset $E(G)$. Given an edge $e$ of $G$ we will often write $G - e$, instead of $G - \{e\}$ to mean the (hyper)graph $G$ with the edge $e$ removed. We let $[r]$ denote the set $\{1, \dots, r\}$.

We now introduce signal senders- these are graphs that are very useful in the construction of minimal graphs. They were introduced in [7] and generalized in [10]. We will only need them in the case that $H = K_k$ but it is not much harder to do so, so we prove their existence for a more general graph.

**Definition 2.1.** For a graph $H$, a *negative signal sender* $S = S^-(r, H, e, f)$ is a graph containing distinguished edges $e$ and $f$ with the following properties:

(i) $S$ is not $r$-ramsey with respect to $H$.
(ii) For every $r$-colouring of $S$ in which there is no monochromatic copy of $H$, edges $e$ and $f$ have different colours.

A *positive signal sender* $S = S^+(r, H, e, f)$ is the same but we replace the word 'different' in property (ii) with 'the same'.

In [7] it was proved that there exist positive and negative signal senders for complete graphs when $r = 2$. In [10] a similar proof was used to extend the result to any 3-connected graph $H$ when $r = 2$.

We extend the result of [10] to all $r \geq 2$ with the following result.

**Lemma 2.2.** *For $r \geq 2$ and 3-connected graph $H$ there exist*

a) *negative signal senders $S^-(r, H, e, f)$, and*
b) *positive signal senders $S^+(r, H, e, f)$.*
c) *Moreover, there exist such senders in which the distance between the edges $e$ and $f$ is arbitrarily large.*

The proof of this Lemma is similar to the proof in [10]. We include it in Section 7 for completeness.

Lemma 2.2, and the Construction in Section 3.1 are the main tools that we will use in the proof of Theorem 1.1.

## 3. Theorem 1.1: The Base Case

In this section we show that there exist $n_0(2, 3)$ and $c = c(2, 3) > 1$ such that for any $n \geq n_0(2, 3)$ there exist at least $c^{n^2}$ non-isomorphic graphs on at most $n$ vertices that are 2-ramsey-minimal with respect to the triangle, $K_3$. This will be the base case for the inductions we use in the proof of Theorem 1.1.

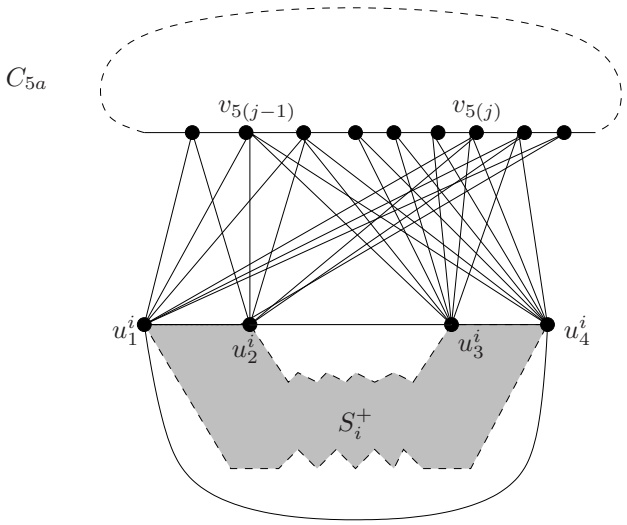


FIGURE 1. Edges of $E_i$ and $F_i$ when $j-1, j+1 \in L_i$ and $j \in R_i$.

Let $a$ be an odd integer. For $i = 1, \ldots, a$, let $L_i \cup R_i = [a]$ be a partition. Let $\mathcal{P} = \{L_i \cup R_i \mid i = 1, \ldots, a\}$. For each of the $2^{a^2}$ such choices of $\mathcal{P}$ we will construct a graph $G(\mathcal{P})$ that is 2-ramsey-minimal with respect to the triangle. We will use this construction in Theorem 3.12 to prove our main result for the case $r = 2$ and $k = 3$.

The graph $G = G(\mathcal{P})$ will be built from three components: $G^*(\mathcal{P}), G^\uparrow$, and $G^\downarrow$. $G^*(\mathcal{P})$ is built in Section 3.1, and $G^\uparrow$, and $G^\downarrow$ are built in Section 3.2. In Section 3.3 these graphs are used to construct $G(\mathcal{P})$.

3.1. **The Graph $G^*(\mathcal{P})$.** For odd $a$ and set $\mathcal{P} = \{L_i \cup R_i \mid i = 1, \ldots, a\}$ of partitions of $[a]$, we construct $G^*(\mathcal{P})$ as follows. (Recall that we will often identify a graph $G$ with its edgeset $E(G)$.)

**Construction 3.1.**

(i) Let $C_{5a}$ be a cycle of length $5a$ with vertices $v_1, \ldots, v_{5a}$ and edges $\{v_j, v_{j+1}\}$ for all $j = 1, \ldots, 5a$ (indices mod $5a$).

(ii) For $i = 1, \ldots, a$, let $S_i^+$ be a copy of the positive signal sender $S^+ = S^+(2, K_3, e, f)$ with distance 3 between edges $e$ and $f$ (c.f. Lemma 2.2). Let $\{u_1^i, u_2^i\}$ and $\{u_3^i, u_4^i\}$ be the copies of $e$ and $f$, respectively, in $S_i^+$. Let $U^i = \{u_1^i, u_2^i, u_3^i, u_4^i\}$

(iii) For $i = 1, \ldots, a$, add the edges $\{u_1^i, u_4^i\}$ and $\{u_2^i, u_3^i\}$. Note that this doesn't introduce any new triangles, because we made the assumption that there is distance 3 between $e$ and $f$ in $S^+$. (See Figure 1.)

(iv) For vertex sets $X$ and $Y$, let $K(X, Y)$ denote the complete bipartite graph between $X$ and $Y$. For each partition $L_i \cup R_i \in \mathcal{P}$, add the following

unions of complete bipartite graphs.

$$E_i = \bigcup_{j\in L_i} K(\{u_1^i, u_2^i\}, \{v_{5(j-1)}, \dots, v_{5j+1}\}), \text{ and}$$
$$F_i = \bigcup_{j\in R_i} K(\{u_3^i, u_4^i\}, \{v_{5(j-1)}, \dots, v_{5j+1}\}).$$

(See Figure 1.) Let $E = \bigcup_{i=1}^a E_i$ and $F = \bigcup_{i=1}^a F_i$. Observe that for $j \in [a]$ the set $\{v_{5(j-1)}, \dots, v_{5j+1}\}$ contains 7 vertices, not just 5. Thus the two sets $\{v_{5(j-1)}, \dots, v_{5j+1}\}$ and $\{v_{5(j)}, \dots, v_{5(j+1)+1}\}$ overlap in two vertices.

Denote the resulting graph by $G^*(\mathcal{P})$. Thus

$$V(G^*(\mathcal{P})) = \{v_1, \dots, v_{5a}\} \cup \bigcup_{i=1}^a V(S_i^+), \text{ and} \tag{1}$$
$$G^*(\mathcal{P}) = C_{5a} \cup E \cup F \cup \bigcup_{i=1}^a S_i^+ \cup \bigcup_{i=1}^a \{\{u_1^i, u_4^i\}, \{u_2^i, u_3^i\}\}.$$

**Definition 3.2.** We classify the vertices of $C_{5a}$ in $G^*(\mathcal{P})$ with regard to their adjacencies to the graphs $S_i^+$.

If one of the indices $j-1$ and $j$ are in $L_i$, while the other one is in $R_i$, then the vertices $v_{5j}$ and $v_{5j+1}$ are adjacent to all four vertices of $U^i = \{u_1^i, u_2^i, u_3^i, u_4^i\}$. In this case, we call the vertices $v_{5j}$ and $v_{5j+1}$ *external for* $U^i$.

Other vertices of $C_{5a}$ are adjacent to exactly two of the vertices of $U^i$. We call these vertices *internal for* $U^i$.

An edge from $v_j \in C_{5a}$ to $U^i$ is an *internal edge* if the vertex $v_j$ is internal for $U^i$. An internal edge of the form $e = \{u_1^i, v_{5j+3}\}$ for $i \in \{1, \dots, a\}$ and $j \in \{0, \dots, a-1\}$ is called a *central edge*.

**Lemma 3.3.** *The graph $G^*(\mathcal{P})$ has the following properties.*

(i) $|V(G^*)| = a(5 + |V(S^+)|) < 2a|V(S^+)|$

(ii) *If $\chi : G^* \to \{1, 2\}$ is a 2-colouring of the edges of $G^*$ such that $\chi$ is monochromatic on $C_{5a}$ and $\chi(\{u_1^i, u_2^i\}) \neq \chi(C_{5a})$ for some $i \in \{1, \dots, a\}$, then $\chi$ has a monochromatic triangle.*

(iii) *For any central edge $e$, $G^* - e$ has a triangle-free 2-colouring $\chi$ of its edges such that*
   (a) *$\chi$ is monochromatic on $C_{5a}$, and*
   (b) *$\chi(\{u_1^i, u_2^i\}) = \chi(C_{5a})$ for all edges $\{u_1^1, u_2^1\}, \dots, \{u_1^a, u_2^a\}$ except one.*

*Proof.* Property (i) follows from (1) and the fact that $|V(S^+)| > 5$. The proofs of the other two properties will be considerably longer.

**(ii)** The proof of property (ii) relies on the observation that *the only triangle-free 2-colourings $\chi$ of the complete graph $K_4$ on vertices $a_1, a_2, b_1$ and $b_2$ with $\chi(\{a_1, a_2\}) \neq \chi(\{b_1, b_2\})$ have*

(∗) $\chi(\{a_1, b_1\}) = \chi(\{a_2, b_2\}) \neq \chi(\{a_1, b_2\}) = \chi(\{a_2, b_1\})$.

Let $\chi$ be a triangle-free 2-colouring of $G^*$ such that $\chi$ is monochromatic on $C_{5a}$, and $\chi(\{u_1^i, u_2^i\}) \neq \chi(C_{5a})$ for some $i$. We will assume, wlog, that this $i$ is 1, i.e., that

$$\chi(\{u_1^1, u_2^1\}) \neq \chi(C_{5a}).$$

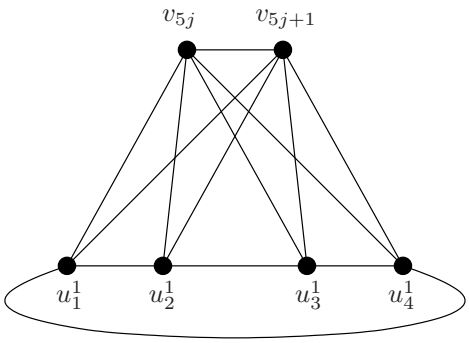


FIGURE 2. Subgraph of $G^*$ induced by $U^1 \cup \{v_{5j}, v_{5j+1}\}$.

By the properties of $S_1^+$, we have that $\chi(\{u_1^1, u_2^1\}) = \chi(\{u_3^1, u_4^1\})$.

The following observations are specific cases of $(*)$.

- $(*_1)$ Edges from any one of the vertices of $U^1$ to two consecutive vertices $v_i$ and $v_{i+1}$, must get different colours.
- $(*_2)$ Edges from $v_i$ to either $u_1^1$ and $u_2^1$, or $u_3^1$ and $u_4^1$ must get different colours.

Indeed, if, say $u_1^1$ were adjacent to both $v_i$ and $v_{i+1}$, then these vertices and $u_2^1$ would induce a $K_4$, and so observation $(*)$ yields $(*_1)$. Similarily, if $v_i$ is adjacent to say $u_1^1$ and $u_2^1$, then so is one of $v_{i-1}$ and $v_{i+1}$. These vertices then induce a $K_4$, and consequently, observation $(*)$ yields $(*_2)$.

The proof of property (ii) of the lemma follows from the following claim.

**Claim 3.4.** *If a vertex $v_i$ of $C_{5a}$ is external for $U^1$, then $\chi(\{u_1^1, v_i\}) = \chi(\{u_3^1, v_i\})$.*

Indeed, assuming this claim, we are done because every $v_i$ in $C_{5a}$ is adjacent to either $u_1^1$ or $u_3^1$, and so by observation $(*_1)$,

$$\chi(\{-, v_1\}) \neq \chi(\{-, v_2\}) \neq \cdots \neq \chi(\{-, v_{5a}\}) \neq \chi(\{-, v_1\}),$$

where '$-$' denotes either $u_1^1$ or $u_3^1$. However, $5a$ is odd, and $\chi$ takes on only two values, so this is a contradiction.

To finish the proof of property (ii), we now prove Claim 3.4.

*Proof.* If the vertex $v_i$ is external for $U^1$ then $i = 5j$ or $5j+1$ where $j \in L_1$ and $j-1 \in R_1$, (or where $j \in R_1$ and $j-1 \in L_1$, but the argument for this case in analogous.) In this situation, all possible edges between $U^1$ and $\{v_{5j}, v_{5j+1}\}$ are in $G^*$. (See Figure 2.) We show that $\chi(\{u_1^1, v_{5j}\}) = \chi(\{u_3^1, v_{5j}\})$. The argument that $\chi(\{u_1^1, v_{5j+1}\}) = \chi(\{u_3^1, v_{5j+1}\})$ is analagous.

Assume, wlog, that $\chi(\{u_1^1, v_{5j}\}) = 1$. By $(*_2)$, $\chi(\{u_2^1, v_{5j}\}) = 2$, and then by $(*_1)$, $\chi(\{u_2^1, v_{5j+1}\}) = 1$. We now have two cases.

- If $\chi(\{u_2^1, u_3^1\}) = 2$, then since $\chi$ is triangle-free, we have that $\chi(\{u_3^1, v_{5j}\}) = 1$. This is the same as $\chi(\{u_1^1, v_{5j}\})$, as needed.
- If $\chi(\{u_2^1, u_3^1\}) = 1$. then since $\chi$ is triangle free, $\chi(\{u_3^1, v_{5j+1}\}) = 2$. Thus by $(*_1)$, $\chi(\{u_3^1, v_{5j}\}) = 1$. This is the same as $\chi(\{u_1^1, v_{5j}\})$, as needed.

□

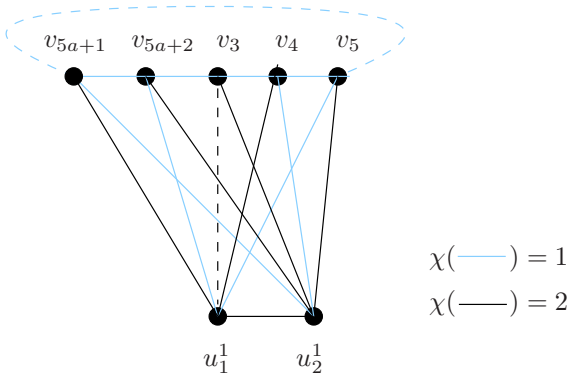


FIGURE 3. The colouring $\chi$.

This completes the proof of the claim, and so completes the proof of (ii).

**(iii)** Without loss of generality, we can assume that the central edge $e$ is $\{u_1^1, v_3\}$. Thus $1 \in L_1$, so $G^*$ contains all edges between $\{u_1^1, u_2^1\}$ and $\{v_2, v_3, v_4\}$, and none of the edges between $\{u_3^1, u_4^1\}$ and $\{v_2, v_3, v_4\}$.

We now define a 2-colouring $\chi$ of $G^* - e$.

(a) Let $\chi(e) = 1$ for all $e \in C_{5a}$.
(b) Let $\chi(\{u_1^1, u_2^1\}) = \chi(\{u_2^1, u_3^1\}) = \chi(\{u_3^1, u_4^1\}) = \chi(\{u_4^1, u_1^1\}) = 2$ and for all $i = 2, \dots a$, let $\chi(\{u_1^i, u_2^i\}) = \chi(\{u_2^i, u_3^i\}) = \chi(\{u_3^i, u_4^i\}) = \chi(\{u_4^i, u_1^i\}) = 1$

Conditions (a) and (b) required by the lemma are thus satisfied, we have now to extend $\chi$ to a triangle-free 2-colouring of $G^*$.

- For all $i \in [a]$ we have $\chi(\{u_1^i, u_2^i\}) = \chi(\{u_3^i, u_4^i\})$, so by the properties of $S_i^+$ we can (and do) extend $\chi$ to a triangle-free 2-colouring of $S_i^+$.
- For $i = 2, \dots, a$ let $\chi$ have colour 2 on all edges between $U^i$ and $V(C_{5a})$

Observe that so far we have no monochromatic triangles. It remains to extend $\chi$ to the edges between $V(C_{5a})$ and $U^1$. This is explained below and illustrated in Figures 3 and 4.

- Relabel $v_1$ as $v_{5a+1}$, and $v_2$ as $v_{5a+2}$. (This gives $v_1$ an even index and $v_2$ an odd index.) With this relabeling in mind, for any edge of the form $e = \{v_i, u_j^1\}$, let $\chi(e) = 1$ if $i + j$ is even, or $\chi(e) = 2$ if $i + j$ is odd.

We now verify that $\chi$ is triangle-free on the subgraph of $G^* - \{u_1^1, v_3\}$, that is induced by $V(C_{5a}) \cup U^1$. This will complete the proof. There are two types of triangles that we must consider.

- We first consider any triangle with an edge induced by $U^1$. Such a triangle must also contain a vertex $v_i \in V(C_{5a}) - \{v_3\}$. However the colours of the edges from $v_i$ to any to adjacent vertices of $U^1$ are different. So such a triangle is not monochromatic.

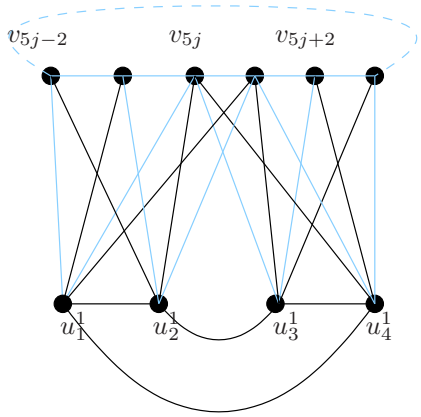


FIGURE 4. The colouring $\chi$ when $j$ is odd, $j-1 \in L_1$ and $j \in R_1$.

- We then consider any triangle with an edge induced by $V(C_{5a}) - \{v_3\}$. Such a triangle must also contain a vertex $u_i^1 \in U^1$. However the colours of the edges from $u_i^1$ to any to adjacent vertices of $V(C_{5a}) - \{v_3\}$ are different. So such a triangle is not monochromatic.

We have thus shown that there is a triangle-free 2-colouring $\chi$ of $G^* - \{u_1^1, v_3\}$, that satisfies properties (a) and (b). This completes the proof of the lemma.

□

### 3.2. The Graphs $G^\uparrow$ and $G^\downarrow$.

For positive integer $a \geq 3$ we construct $G(5a)^\uparrow$ as follows.

**Construction 3.5.**

(i) Let $C_{5a}$ be a cycle of length $5a$ with vertices $v_1, \ldots, v_{5a}$ and edges $\{v_i, v_{i+1}\}$ for all $i = 1, \ldots, 5a$.
(ii) For $i = 1, \ldots, 5a$, let $S_i = S^-(2, K_3, e_i, f_i)$ be a copy of the negative signal sender $S^- = S^-(2, K_3, e, f)$ with and distance 2 between edges $e$ and $f$.
(iii) For $i = 1, \ldots, 5a$ identify edge $\{v_i, v_{i+1}\} \in C_{5a}$ (indices mod $5a$) with edge $e_i \in S_i$.
(iv) Identify the $5a$ edges $f_i \in S_i$ for all $i = 1, \ldots, 5a$. Denote the resulting edge by $f$.

Unless explicity identified, all vertices in the construction are distinct.

**Lemma 3.6.** *For integer $a \geq 3$, the graph $G^\uparrow = G^\uparrow(5a)$ has the following properties.*

(i) $|V(G^\uparrow)| < 5a|V(S^-)|$
(ii) *$C_{5a}$ is monochromatic under any triangle free 2-colouring of $G^\uparrow$, and such a colouring exists.*

*Proof.* The property that $|V(G^\uparrow)| < 5a|V(S^-)|$ follows from the fact that every vertex of $G^\uparrow$ is in at least one of the $5a$ copies of $S^-$. We now verify the second property.

Without loss of generality, let $\chi(f) = 2$. Now $\chi$ can be extended to a triangle free colouring of each of the $S_i$. This done, we necessarily have that all of the edges

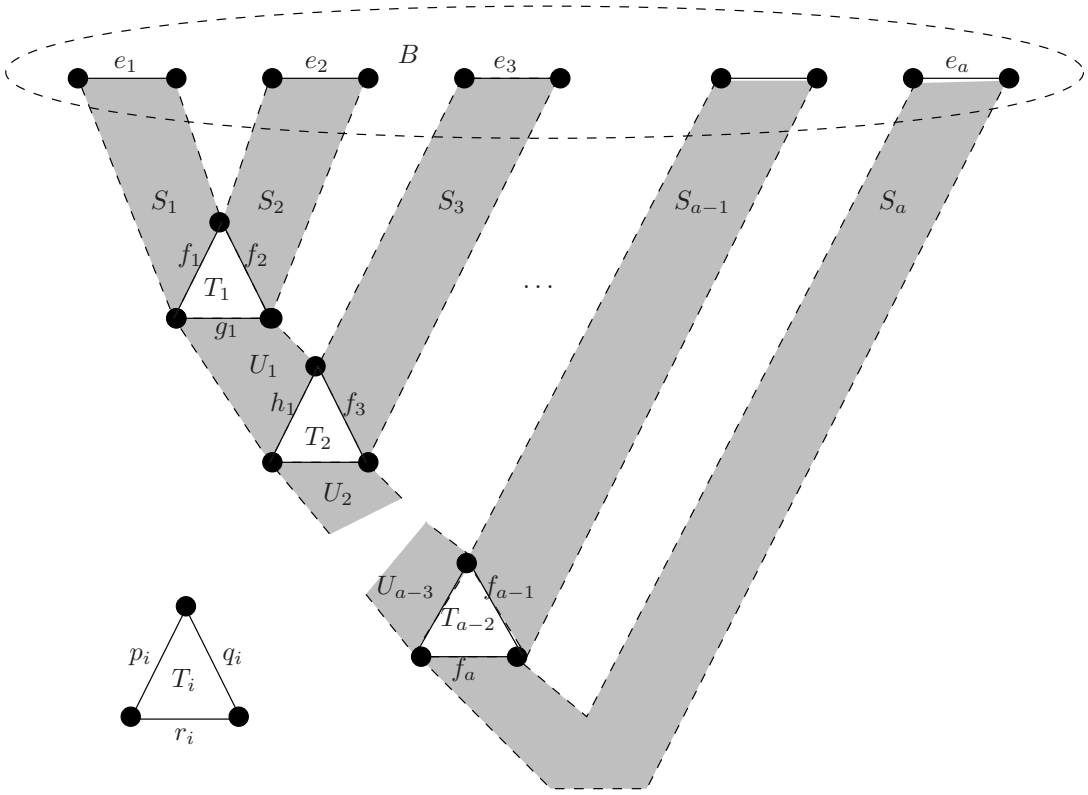


FIGURE 5. $G^{\downarrow}(a)$.

$e_i$, for $i = 1, \dots 5a$, get colour 1, i.e., $C_{5a}$ is monochromatic. By the assumption that edges $e$ and $f$ are distance 2 apart the only triangles in $G^{\uparrow}$ are entirely within one copy of $S^-$. Thus the colouring $\chi$ is a triangle-free colouring of $G^{\uparrow}$. This gives property (ii), so completes the proof of the lemma. □

For integer $a \geq 6$, we construct the graph $G^{\downarrow}(a)$ as follows. (See Figure 5).

**Construction 3.7.**

(i) Let $B$ be a set of $a$ pairwise vertex disjoint edges, $B = \{e_1, \dots, e_a\}$. ( In Construction 3.9, $B$ will be identified with the set $\{\{u_1^i, u_2^i\} \mid i = 1, \dots, a\}$.)
(ii) For integer $i = 1, \dots, a$, let $S_i$ be a copy of $S^- = S^-(2, K_3, e, f)$, and let $e_i$ and $f_i$ be the copies of $e$ and $f$ in $S_i$ respectively.
(iii) For integer $i = 1, \dots, a-3$ let $U_i$ be a copy of $S^-$, and let $g_i$ and $h_i$ be the copies of $e$ and $f$ in $U_i$ respectively.
(iv) For integer $i = 1, \dots, a-2$, let $T_i$ be a triangle with edges $p_i$, $q_i$, and $r_i$.
(v) Identify edges as in Figure 5.

**Lemma 3.8.** *For integer $a \geq 3$, the graph $G^{\downarrow} = G^{\downarrow}(a)$ has the following properties.*

(i) $|V(G^{\downarrow})| < (2a-3)|V(S^-)|$
(ii) *There is no triangle free 2-colouring of $G^{\downarrow}$ that is monochromatic on $B$.*
(iii) *Any 2-colouring $\chi$ of $B$, in which one of the colour classes contains a single edge, can be extended to a triangle free 2-colouring of $G^{\downarrow}$.*

*Proof.* Property (i) follows from the fact that every vertex of $G^{\downarrow}$ occurs in at least one of the $2a-3$ copies $S_i$ or $U_i$ of $S^-$.

To verify property (ii), we must show that every 2-colouring of $G^{\downarrow}$ which is monochromatic on $B$ must contain a monochromatic triangle.

Let $\chi : G^{\downarrow} \to \{1,2\}$ be a 2-colouring of the edges of $G^{\downarrow}$ such that $\chi(B) = 1$. Assume that each sender $S_i$ or $U_i$ in $G^{\downarrow}$ is triangle free under $\chi$. We show that at least one of the triangles in the the set $\mathcal{T} = \{T_1, \ldots, T_{a-2}\}$ must be monochromatic by showing that $2|\mathcal{T}| + 1 = 2a - 3$ of the $3(a-2)$ edges making up the triangles $\mathcal{T}$ get colour 2 under $\chi$.

Since $\chi(B) = \{1\}$, the properties of $S^-$ give us that $\chi(f_i) = 2$ for $i = 1, \ldots, a$. These $f_i$ are distinct edges of the triangles $\mathcal{T}$. Furthermore, for each $i = 1, \ldots, a-3$, $U_i$ is a negative signal sender under a triangle-free colouring, so either $\chi(g_i) = 2$ or $\chi(h_i) = 2$. Again, these are all distinct edges of the triangles $\mathcal{T}$. Thus $a + (a-3) = 2a - 3$ of the edges of the triangles $\mathcal{T}$ are colour 2. One of the triangles must therefore be monochromatic, and the property is verified.

For property (iii), assume that $\chi$ has colour 1 on all edges of $B$ except for the $i^{th}$ one (i.e., the one in $S_i$). If $1 < i < a$, then $\chi$ can be extended to a triangle free colouring of $G^{\downarrow}$ such that

- for $j < i-1$, $\chi(p_j) = \chi(q_j) = 2$ and $\chi(r_j) = 1$;
- for $j = i-1$, $\chi(p_j) = \chi(r_j) = 2$ and $\chi(q_j) = 1$; and
- for $j > i-1$, $\chi(p_j) = 1$ and $\chi(q_j) = \chi(r_j) = 2$.

If $i = 1$ then $\chi$ can be extended to a triangle free colouring of $G^{\downarrow}$ such that for all $j = 1, \ldots, a-2$, $\chi(p_j) = 1$ and $\chi(q_j) = \chi(r_j) = 2$.

The case that $i = a$ is symmetric to the case $i = 1$.

□

### 3.3. **Construction of $G(\mathcal{P})$.**

Given a set $\mathcal{P} = \{L_i \cup R_i \mid i = 1, \ldots, a\}$ of partitions of the set $\{1, \ldots, a\}$, we first define the graph $G'(\mathcal{P})$. The graph $G(\mathcal{P})$ will then be a 2-ramsey-mininal subgraph of this graph.

We construct $G'(\mathcal{P})$ as follows.

**Construction 3.9.**

(i) Let $G^*$ be a copy of $G^*(\mathcal{P})$.
(ii) Let $G^{\uparrow}$ be a copy of $G^{\uparrow}(5a)$.
(iii) Let $G^{\downarrow}$ be a copy of $G^{\downarrow}(a)$.
(iv) Identify the cycle $C_{5a}$ in $G^*$ with the cycle $C_{5a}$ in $G^{\uparrow}$. For $i = 1, \ldots, a$, set $e_i = \{u_1^i, u_2^i\}$; i.e., identify the set $\{\{u_1^i, u_2^i\} \mid i = 1, \ldots, a\}$ of pairwise vertex-disjoint edges in $G^*$ with set $B = \{b_i \mid i = 1, \ldots, a\}$ of pairwise vertex-disjoint edges in $G^{\downarrow}$.

Denote the resulting graph by $G'(\mathcal{P})$.

**Lemma 3.10.** *The graph $G' = G'(\mathcal{P})$ has the following properties.*

(i) *Any 2-colouring of $G'$ has a monochromatic triangle.*
(ii) *For any central edge $e$ of $G^* \subset G'$ there exists a triangle-free 2-colouring of $G' - e$.*

*Proof.*

(i) Assume there is a triangle free 2-colouring of $G'$. Then by Lemmas 3.6(ii) and 3.8(ii), $C_{5a}$ is monochromatic and $B$ is not. Consequently, by Lemma 3.3(ii), $G^*$ in $G'$ contains a monochromatic triangle.
(ii) Let $e$ be a central edge of $G^*$. By Lemma 3.3(iii), $G^* - e$ can be coloured with $C_{5a}$ monochromatic of colour 1, and with $B$ getting colour 1 on all edges but one. Thus Lemmas 3.6(ii) and 3.8(iii) guarantee that the rest of the $G' - e$ can be coloured without any monochromatic triangles.

□

Let $G(\mathcal{P})$ be any subgraph of $G'(\mathcal{P})$ that is minimal with respect to property (i) of Lemma 3.10. By propery (ii) of Lemma 3.10, $G(\mathcal{P})$ must contain all the central edges of $G^*$. Thus we have the following

**Fact 3.11.** *For distinct sets $\mathcal{P}$ and $\mathcal{P}'$, the graphs $G(\mathcal{P})$ and $G(\mathcal{P}')$ are different.*

Let $S^+$ be the positive sender $S^+(2, K_3, \ell, r)$ used in the construction of $G^*(\mathcal{P})$, and $S^-$ be the negative sender $S^-(2, K_3, e, f)$ used in the constructions of $G^\uparrow$ and $G^\downarrow$. Let $s = \max(|V(S^+)|, |V(S^-)|)$. We then have the following $r = 2$, $k = 3$ case of Theorem 1.1.

**Theorem 3.12.** *There exists a constant $c = c(2,3) > 1$ such for $n \geq n_0 = n_0(2,3) = 99s$ there exist at least $c^{n^2}$ non-isomorphic graphs, on at most $n$ vertices, that are 2-ramsey-minimal with respect to the triangle.*

*Proof.* Let $a$ be the largest odd integer for which $a \leq \frac{n}{9s}$. Since $n \geq 99s$, $a \geq \frac{n}{9s} - 2 \geq 11 - 2 = 9$, which yields that

$$a \geq \frac{9}{11}\frac{n}{9s} = \frac{n}{11s}.$$

Now there are $2^{a^2}$ ways to choose the set $\mathcal{P}$ of $a$ partitions $L_i \cup R_i$ of $\{1, \ldots, a\}$. By Fact 3.11, each $\mathcal{P}$ yields a different graph $G(\mathcal{P})$. We therefore have $2^{a^2}$ different graphs $G(\mathcal{P})$. Using part (i) of Lemmas 3.3, 3.6, and 3.8, we get that each has

$$\begin{aligned} |V(G(\mathcal{P}))| &< |V(G^*(\mathcal{P}))| + |V(G^\uparrow(5a))| + |V(G^\downarrow(a))| \\ &< 2as + 5as + 2as \\ &= 9as \leq n \end{aligned}$$

vertices.

Setting $c_1 = 1/(11s)^2$, there are $2^{a^2} > 2^{c_1 n^2}$ different such labelled graphs. The isomorphism class of any one is at most $n!$, so there are at least

$$\frac{2^{c_1 n^2}}{n!} = 2^{c_1 n^2 (1 + o(1))} > 2^{c_2 n^2},$$

non-isomorphic graphs, for $c_2 < c_1$, and $n$ sufficiently large.

This is $c^{n^2}$, where $c = 2^{c_2}$, so the proof is complete. □

## 4. Theorem 1.1: Induction on $k$

We prove Theorem 1.1 for $r = 2$ and all $k$, by induction on $k$. The induction will be done by means of the following lemma. The proof of the lemma takes up the majority of the section. The proof of the induction on $k$ comes only at the very end on the section.

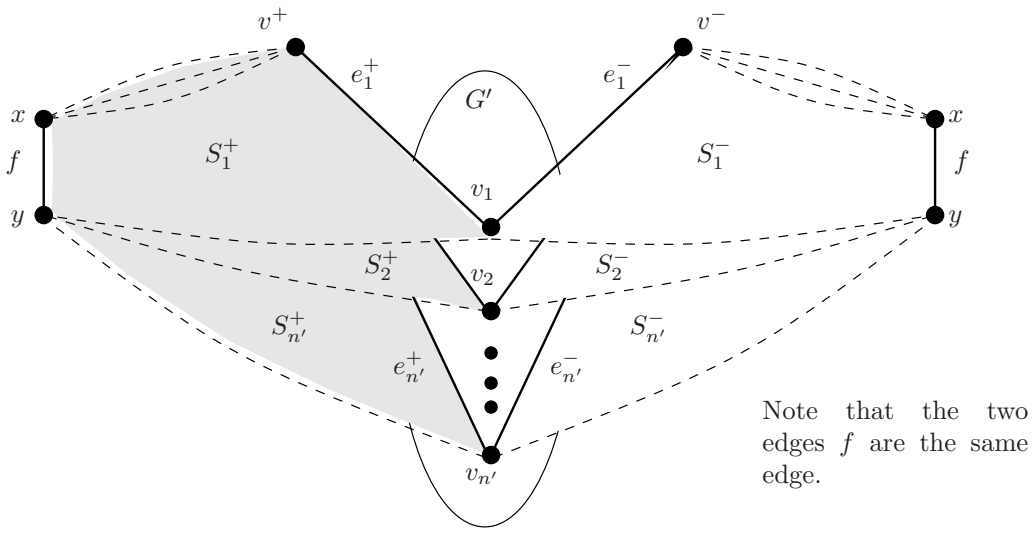


FIGURE 6. Construction of $G$ from Lemma 4.1.

**Lemma 4.1.** *Given an integer $k$ there exists a constant $s = s(k)$ such that for a graph $G'$ on $n'$ vertices that is 2-ramsey-minimal with respect to $K_{k-1}$, there exists a graph $G$ with the following properties.*

(i) *The graph $G$ has at most $sn'$ vertices,*
(ii) *$G$ is 2-ramsey with respect to $K_k$, and*
(iii) *any 2-ramsey-minimal subgraph of $G$ contains $G'$ as a subgraph.*

*Proof.* Let $V(G') = \{v_1, \dots, v_{n'}\}$, and construct $G$ as follows. (See Figure 6.)

- For $i = 1, \dots, n'$ let $S_i^- = S^-(2, K_k, e_i^-, f_i^-)$, and $S_i^+ = S^+(2, K_k, e_i^+, f_i^+)$ be new copies of the senders $S^-(2, K_k, e, f)$ and $S^+(2, K_k, e, f)$ respectively. Assume that in all of these senders, the copies of $e$ and $f$ are distance at least 2 apart.
- Add new vertices $v^-$ and $v^+$. For each $i = 1, \dots n'$, identify the endpoints of $e_i^-$ with $v^-$ and $v_i$, and the endpoints of $e_i^+$ with $v^+$ and $v_i$.
- Identify all of the $f_i^+$ and $f_i^-$ and call this edge $f$. Let $x$ and $y$ be the endpoints of $f$.

Set $s = |V(S^-(2, K_k, e, f))| + |V(S^+(2, K_k, e, f))|$. We now verify that the graph $G$, thus constructed, has all the properties promised by the lemma.

To see property (i) notice that every vertex of $G$ is in at least one of the $n'$ copies of $S^-(2, K_k, e, f)$ or the $n'$ copies of $S^+(2, K_k, e, f)$. Thus $|V(G)| < n'(|V(S^-(2, K_k, e, f))| + |V(S^+(2, K_k, e, f))|) = sn'$.

Property (ii) will be verified in Claim 4.2, and property (iii) follows immediately from Claim 4.4.

**Claim 4.2.** *Any 2-colouring of $G$ yields a monochromatic $K_k$.*

*Proof.* Let $\chi$ be a 2-colouring of $G$ and assume that without loss of generality that $\chi(f) = 1$. Then by the properties of the $S_i^-$, all edges from $v^-$ to $G'$ get colour 2, and by the properties of the $S_i^+$, all edges from $v^+$ to $G'$ get colour 1. Now $\chi$ induces a 2-colouring of $G'$, which yields a monochromatic copy of $K_{k-1}$ in $G'$. If

this $K_{k-1}$ is colour 1, then it makes a monochromatic $K_k$ with $v^+$. Otherwise, it makes a monochromatic $K_k$ with $v^-$. □

The following claim is used in Claim 4.4.

**Claim 4.3.** *Any copy $K'_k$ of $K_k$ in $G$ either intersects $G'$ in at least a $K_{k-1}$, or occurs within $S_i^-$ or $S_i^+$ for some $i = 1, \dots, n'$.*

*Proof.* We split the proof of the claim into three cases.

**Case 1:** $|V(K'_k) \cap V(G')| \geq 2$.
Without loss of generality, let $v_1$ and $v_2$ be in $|V(K'_k) \cap V(G')|$. Recall that $x$ and $y$ are the endpoints of $f$. By the assumption that the edges $e$ and $f$ in $S^-(2, K_k, e, f)$ and $S^+(2, K_k, e, f)$ are distance at least 2 apart, the only neighbours of $v_1$ are in $V(G') \cup V(S_1^+) \cup V(S_1^-) - \{x, y\}$. (Recall that $x$ and $y$ are the endpoints of $f$.) Similarly the only neighbours of $v_2$ are in $V(G') \cup V(S_2^+) \cup V(S_2^-) - \{x, y\}$. Thus the only common neighbours of $v_1$ and $v_2$ are in $V(G') \cup \{v^-, v^+\}$. Since the vertices $v^+$ and $v^-$ are non-adjacent, only one of them belongs to $V(K'_k)$. Consequently, $|V(K'_k) \cap V(G')| \geq k - 1$, i.e. $K'_k$ intersects $G'$ in at least a $K_{k-1}$.

**Case 2:** $|V(K'_k) \cap V(G')| = 1$.
Without loss of generality, let $V(K'_k) \cap V(G') = \{v_1\}$. The only neighbours of $v_1$ are in $S_1^-$ or $S_1^+$, but there are no edges between $V(S_1^-)$ and $V(S_1^+)$ except those incident to $v_1$. Consequently, $K'_k$ occurs within $S_1^-$ or $S_1^+$.

**Case 3:** $|V(K'_k) \cap V(G')| = 0$.
Since the set $\{v^-, v^+, x, y\}$ induces only the one edge $f$, at most 2 of the vertices in this set can be in $K'_k$. Therefore there is some other vertex $v$ of one of the $S_i^-$ or $S_i^+$ in $K'_k$. Without loss of generality, assume that $v \in S_1^-$. All the neighbours of $v$ are in $S_1^-$. Consequently $K'_k$ occurs within $S_1^-$. □

It follows from the following claim that any 2-ramsey-minimal subgraph of $G$ when restricted to $V(G')$, is $G'$.

**Claim 4.4.** *Removing any edge $e' \in G' \subset G$ we can 2-colour $G$ with no monochromatic $K_k$.*

*Proof.* Removing edge $e'$ from $G'$, we can 2-colour $G'$ without any monochromatic $K_{k-1}$. Since the senders can be 2-coloured without any monochromatic $K_k$, Claim 4.3 implies that this gives a 2-colouring of $G$ without any monochromatic $K_k$. □

This completes the proof of the lemma.

□

We now prove the induction on $k$, thus proving the $r = 2$ case of Theorem 1.1. By Section 3, we have the base case, $k = 3$.

*Proof.* Assume that Theorem 1.1 is true for $r = 2$ and $k - 1$. We must provide integer $n_0(2, k)$ and constant $c(2, k) > 1$ such that for any $n \geq n_0(2, k)$ there exist at least $c(2, k)^{n^2}$ non-isomorphic graphs on at most $n$ vertices, each of which is 2-ramsey-minimal with respect to $K_k$.

Letting $s = s(k)$ be from Lemma 4.1, set

$$c(2,k) = c(2,k-1)^{1/(8s^2)}.$$

Observe that because $c(2,k-1) > 1$, we have that $c(2,k) > 1$. Let $n_0(2,k)$ be large enough that

$$n_0(2,k) \geq s \cdot n_0(2,k-1),$$

and that for all $n \geq n_0(2,k)$,

$$n! \leq c(2,k)^{n^2}. \tag{2}$$

Given

$$n \geq n_0(2,k) \geq s \cdot n_0(2,k-1),$$

set $n' = \lfloor \frac{n}{s} \rfloor$. Since $n_0(2,k-1)$ is an integer, this implies

$$n' \geq n_0(2,k-1).$$

We can therefore use the induction hypothesis to get at least $c(2,k-1)^{(n')^2}$ labeled graphs on at most $n'$ vertices each of which is 2-ramsey-minimal with respect to $K_{k-1}$. Apply Lemma 4.1 to each of these graphs to get graphs on at most $sn' \leq n$ vertices which are 2-ramsey with respect to $K_k$.

Since $n_0(2,k-1) \geq n_0(2,3) = 99$ by Theorem 3.12, so $n \geq n_0(2,k) \geq 99s$ we have that

$$n' = \left\lfloor \frac{n}{s} \right\rfloor > \frac{n}{s} - 1 > \frac{n}{2s},$$

thus there are at least

$$c(2,k-1)^{(n')^2} > c(2,k-1)^{\frac{n^2}{(2s)^2}} = c(2,k)^{2 \cdot n^2}$$

such graphs. By Lemma 4.1 (iii) the 2-ramsey-minimal subgraphs of these graphs are distinct, though not necessarily non-isomorphic. Dividing by $n!$, which is at most $c(2,k)^{n^2}$ by assumption (2), we get at least $c(2,k)^{n^2}$ non-isomorphic 2-ramsey-minimal graphs for $K_k$, each on at most $n$ vertices. This completes the proof of Theorem 1.1 for case $r = 2$. □

## 5. Theorem 1.1: Induction on $r$

In the previous section we proved the $r = 2$ case of Theorem 1.1 by induction on $k$. In this section we finish the proof of the theorem by induction on $r$. We do the induction by means of the following lemma.

**Lemma 5.1.** *Given integers $r$ and $k$ there exists a constant $s = s(r,k)$ such that for a graph $G'$ on $n'$ vertices that is $(r-1)$-ramsey-minimal with respect to $K_k$, there exists a graph $G$ with the following properties.*

(i) *The graph $G$ has at most $sn'$ vertices,*
(ii) *$G$ is $r$-ramsey with respect to $K_k$, and*
(iii) *any $r$-ramsey-minimal subgraph of $G$ contains $G'$ as a subgraph.*

The the induction on $r$ uses Lemma 5.1 in the same way as the induction on $\ell$ used Lemma 4.1, in the previous section.

Let $c(r,k) = c(r-1,k)^{1/8s^2}$. For large enough $n$ we will find at least $c(r,k)^{n^2}$ graphs on less than $n$ vertices that are $r$-ramsey minimal for $K_k$.

By induction, if $n' = \lfloor n/s \rfloor$ is large enough, there are at least $c(r-1,k)^{n'^2}$ graphs on at most $n'$ vertices that are $(r-1)$-ramsey minimal for $K_k$. Applying Lemma 5.1 to each, we get

$$c(r-1,k)n'^2 > c(r-1,k)^{\frac{n^2}{(2s)^2}} = c(r,k)^{2\cdot n^2}$$

distinct labelled graphs on at most $n$ vertices that are $r$-ramsey-minimal for $K_k$. Dividing by $n!$ we get at least $c(r,k)^{n^2}$ non-isomorphic graphs on at most $n$ vertices (for $n$ sufficiently large). This completes the induction step.

We thus finish the proof of Theorem 1.1 by proving Lemma 5.1.

*Proof.* (of Lemma 5.1)

Briefly, $G$ is as follows. Consider $G'$, a copy of $K_{k-2}$, and an edge $f$, which are pairwise vertex disjoint. Add edges between every vertex of $G'$ and every vertex of $K_{k-2}$. Finally, for all edges $e$ with at least one vertex in $V(K_{k-2})$, connect $e$ to $f$ with positive sender $S^+(r, K_k, e, f)$. (Thus $G$ is constructed with $n' \cdot (k-2) + \binom{k-2}{2}$ copies of $S^+(r, K_k, e, f)$.) In more detail, $G$ is as follows.

- Begin with $G'$. Let $V(G') = \{v_1, \dots, v_{n'}\}$.
- Add new vertices $w_1, \dots, w_{k-2}$, all edges between them, and one more vertex disjoint edge $f = \{x, y\}$.
- For $i = 1, \dots, n'$, and $j = 1, \dots, k-2$ add edge $e_{ij} = \{v_i, w_j\}$ and let $S_{ij} = S^+(r, K_k, \{v_i, w_j\}, f)$ be a copy of $S^+(r, K_k, e, f)$ in which all vertices are distinct except for $v_i, w_j, x$, and $y$. Assume that in all of these senders, the copies of $e$ and $f$ are distance at least 2 apart.
- For each $i, j$ such that $1 \le i < j \le k-2$, let $W_{ij} = S^+(r, K_k, \{w_i, w_j\}, f)$ be a copy of $S^+(r, K_k, e, f)$ in which all vertices are distinct except for $w_i, w_j, x$, and $y$. Assume again that in all of these senders, the copies of $e$ and $f$ are distance at least 2 apart.

Set $s = 2k|V(S^+(r, K_k, e, f))|$. We now verify that the graph $G$, constructed above, has all the properties promised by the lemma.

To see property (i) notice that every vertex of $G$ is in at least one of the $n' \cdot (k-2) + \binom{k-2}{2}$ copies of $S^+(r, K_k, e, f)$ in $G$. Since $G'$ has $n'$ vertices, and is ramsey for $K_k$, $n' \ge k$. Thus $\binom{k-2}{2} < n' \cdot k$, so

$$|V(G)| < n'(2k)|V(S^+(r, K_k, e, f))| = sn'.$$

The following claim verifies property (ii).

**Claim 5.2.** *Any $r$-colouring of $G$ yields a monochromatic $K_k$.*

*Proof.* Let $\chi : V(G) \to \{1, \dots, r\}$ be an $r$-colouring of $G$ and assume that without loss of generality that $\chi(f) = r$. If there is no monochromatic $K_k$ in any of the $S_{ij}$ or $W_{ij}$, then by the properties of these senders, all edges $\{v_i, w_j\}$ and $\{w_i, w_j\}$ get colour $r$. If any edge in $G'$ gets coloured $r$, then this completes a monochromatic $K_k$ with vertices $w_1, \dots, w_{k-2}$. Thus $\chi$ must induce an $(r-1)$-colouring on $G'$. But $G'$ is $(r-1)$-ramsey with respect to $K_k$, so $G$ contains a monochromatic $K_k$. □

Property (iii) follows from Claim 5.4. However, for this, we first need the following claim.

**Claim 5.3.** *For any copy $K'_k$ of $K_k$ in $G$, either $V(K'_k) \subset V(G') \cup \{w_1, \cdots, w_{k-2}\}$, or $V(K'_k) \subset V(S^+)$ for some sender $S^+ = S^+_{ij}$ or $W^+_{ij}$.*

*Proof.* First observe that if there exists $v \in V(K'_k)$ such that

$$v \in V(S^+_{ij}) - \{v_i, w_j, x, y\} \qquad \text{or} \qquad v \in V(W^+_{ij}) - \{w_i, w_j, x, y\}, \tag{3}$$

then all the neighbours of $v$ are in $S^+_{ij}$ or $W^+_{ij}$ respectively, thus $V(K'_k) \subset V(S^+_{ij})$ (or $V(W^+_{ij})$), so we are done.

We may thus assume that (3) does not hold for any $v \in V(K'_k)$. It follows that $V(K'_k) \subset V(G') \cup \{w_1, \dots, w_{k-2}\} \cup \{x, y\}$. Since $x$ and $y$ have distance at least 2 from $V(G') \cup \{w_1, \dots, w_{k-2}\}$, and $k > 3$, we have that $V(K'_k) \cap \{x, y\} = \emptyset$. Thus $V(K'_k) \subset V(G') \cup \{w_1, \dots, w_{k-2}\}$, which finishes the proof of the claim. □

The following claim implies that any $r$-ramsey-minimal subgraph of $G$ contains all the edges of $G'$, thus property (iii) of the lemma follows.

**Claim 5.4.** *For any edge $e$ of $G'$, we can $r$-colour $G - e$ with no monochromatic $K_k$.*

*Proof.* Let $e$ be in $G'$, we can $(r-1)$-colour $G' - e$ with colours $\{1, \dots, r-1\}$ without any monochromatic $K_k$. We extend this $(r-1)$-colouring of $G' - e$ to an $r$-colouring of $G - e$. Let all edges $\{v_i, w_j\}$ and $\{w_i, w_j\}$, and the edge $f$, get colour $r$. For each positive sender $S^+_{ij}$ (or $W^+_{ij}$), the copy $\{v_i, w_j\}$ (resp. $\{w_i, w_j\}$) of $e$ gets the same colour as $f$, so we can extend our colouring to a $r$ colouring of $S^+_{ij}$ (or $W^+_{ij}$) with no monochromatic $K_k$. By Claim 5.3, this gives a $K_k$-free $r$-colouring of $G - e$. □

This completes the proof of Lemma 5.1, and by the comments at the beginning of the section, completes the proof of Theorem 1.1 □

## 6. The Non-symmetric Case

In this section we prove the following non-symmetric case of our main theorem.

**Theorem 6.1.** *For any integers $\ell \geq k \geq 3$ there exist constants $c = c(\ell, k) > 1$ and $n_0(\ell, k)$ such that for all $n \geq n_0(\ell, k)$, there exist at least $c^{n^2}$ non-isomorphic graphs, on at most $n$ vertices, that are ramsey-minimal with respect to $K_\ell$ and $K_k$.*

The proof of this theorem uses the following lemma.

**Lemma 6.2.** *Given integers $\ell > k \geq 3$ there exists a constant $s = s(\ell, k)$ such that for a graph $G'$ on $n'$ vertices that is 2-ramsey-minimal with respect to $K_{\ell-1}$ and $K_k$, there exists a graph $G$ with the following properties.*

(i) *The graph $G$ has at most $sn'$ vertices,*
(ii) *$G$ is 2-ramsey with respect to $K_\ell$ and $K_k$, and*
(iii) *any 2-ramsey-minimal subgraph of $G$ contains $G'$ as a subgraph.*

The proof of Theorem 6.1 follows by induction on $\ell$ using Lemma 6.2 in the same way as the $r = 2$ case of Theorem 1.1 followed by induction on $k$ using Lemma 4.1 in Section 4. The base case of the induction, when $\ell = k$, is given by Theorem 1.1.

Let $c(\ell, k) = c(\ell-1, k)^{1/8s^2} > 1$, and $n_0(\ell, k) = s \cdot n_0(\ell-1, k)$. For $n \geq n_0(\ell, k)$ we will find at least $c(\ell, k)^{n^2}$ graphs on less than $n$ vertices that are 2-ramsey minimal for $K_\ell$ and $K_k$.

Given $n \geq n_0(\ell, k)$, let $n' = \lfloor n/s \rfloor \geq n_0(\ell - 1, k)$. By induction, there are at least $c(\ell-1,k)^{n'^2}$ graphs that are 2-ramsey minimal for $K_{\ell-1}$ and $K_k$ on at most $n'$ vertices. Applying Lemma 6.2 to each, we get

$$c(\ell-1,k)n'^2 > c(\ell-1,k)^{\frac{n^2}{(2s)^2}} = c(\ell,k)^{2\cdot n^2}$$

distinct labelled graphs on at most $n$ vertices that are 2-ramsey-minimal for $K_\ell$ and $K_k$. Dividing by $n!$ we get at least $c(\ell,k)^{n^2}$ non-isomorphic graphs on at most $n$ vertices (for $n$ sufficiently large). This completes the induction step.

We thus finish the proof of Theorem 6.1 by proving Lemma 6.2.

*Proof.* (of Lemma 6.2).

First we have to introduce a result from [10].

**Definition 6.3.** Given graphs $H_1$ and $H_2$, a graph $D = D(H_1, H_2, e)$ is an $(H_1, H_2)$-*determiner* if

(i) there is a 2-colouring $\chi : D \to \{1, 2\}$ of $D$ such that there is no copy of $H_1$ of colour 1, and no copy of $H_2$ of colour 2, and
(ii) in any such colouring $\chi$ of $D$, $\chi(e) = 1$.

Theorem 2.3 of [10] states that for any two integers $\ell > k \geq 3$, there exists a $(K_\ell, K_k)$-determiner $D = D(\ell, k, e)$. With this we can prove the lemma.

Given a graph $G'$ as described in the lemma, let $V(G') = \{v_1, \ldots, v_{n'}\}$, and construct $G$ as follows.

- For $i = 1, \ldots, n'$, let $D_i = D(\ell, k, e_i)$ be a copy of the $(K_\ell, K_k)$-determiner $D$.
- Add new vertex $v_0$. For each $i = 1, \ldots n'$, identify the endpoints of $e_i$ with $v_0$ and $v_i$.

The verification that the graph $G$ constructed above has the necessary properties of the lemma is very similar to the proof of Lemma 4.1. □

## 7. Signal Senders

Here we give the proof of Lemma 2.2, which claimed the existence of negative and positive signal senders for all 3-connected graphs $H$. The proof requires some definitions and preliminary results. The first definitions are well known for an $\ell$-graph, i.e. a system $\mathcal{H}$ of $\ell$ element subsets of a vertex set $V(\mathcal{H})$. We will assume that an $\ell$-graph $\mathcal{H}$ has no independent vertices, i.e. that $V(\mathcal{H}) = \cup_{e\in\mathcal{H}} e$.

**Definition 7.1.** A *circuit* of length $h$ in an $\ell$-graph $\mathcal{H}$ is a subgraph $\{e_1, \ldots, e_h\} \subset \mathcal{H}$ such that

$$|\bigcup_{i=1}^{h} e_i| \leq (\ell - 1)h.$$

The *girth* of an $\ell$-graph is the length of its shortest circuit.

Note that this is one of several different notions of a circuit in an $\ell$-graph.

**Definition 7.2.** An $\ell$-graph is $(r+1)$-*chromatic* if $r+1$ is the minimum number of colours needed to colour its vertices such that there is no edge with all vertices having the same colour.

The following is proved probabilistically in [11], and constructively in [14] and [17].

**Theorem 7.3.** *Given integers $g, r \geq 2$ and $\ell \geq 3$ there exists $(r+1)$-chromatic $\ell$-graph $\mathcal{G}$ of girth $\geq g$.*

We will need the following technical definitions.

**Definition 7.4.** Let $r, \ell \geq 2$ be integers.

(i) An *oriented $\ell$-graph* $\boldsymbol{\mathcal{H}}$ on the vertices $V = V(\boldsymbol{\mathcal{H}})$ is a set of ordered $\ell$ element subsets, or arcs, of $V$.

(ii) An $(r,\ell)$-*pattern* is an $\ell$-tuple of not necessarily distinct elements from $[r]$, ( i.e. an element of $\prod_{i=1}^{\ell}[r]$.)

(iii) Given a set $\mathcal{C} \subsetneq \prod_{i=1}^{\ell}[r]$ of forbidden $(r,\ell)$-patterns, an $r$-colouring $\chi : V(\boldsymbol{\mathcal{H}}) \to [r]$ of the vertices of $\boldsymbol{\mathcal{H}}$ is $\mathcal{C}$-*excluded* if $(\chi(v_1), \ldots, \chi(v_\ell)) \notin \mathcal{C}$ for any $(v_1, \ldots, v_\ell) \in \boldsymbol{\mathcal{H}}$.

Observe that if $\mathcal{C} = \{(i, i, \ldots, i) \mid 1 \leq i \leq r\}$ is the set of monochromatic $(r,\ell)$-patterns, then a $\mathcal{C}$-excluded $r$-colouring of an oriented $\ell$-graph $\boldsymbol{\mathcal{H}}$ is simply an $r$-colouring of the vertices of $\boldsymbol{\mathcal{H}}$ in which no edge is monochromatic.

**Lemma 7.5.** *Given integers $g, r, \ell \geq 2$, let $\mathcal{C}$ be a proper subset of the set of all $(r,\ell)$-patterns, which contains the monochromatic pattern $(i, i, \ldots, i)$ for $i = 1, \ldots, r$. Then there exists an oriented $\ell$-graph $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\mathcal{C}, r, \ell, g)$ with girth $\geq g$, having distinguished vertices $u$ and $u'$, satisfying the following.*

(i) *$\boldsymbol{\mathcal{H}}$ has a $\mathcal{C}$-excluded colouring.*

(ii) *The vertices $u$ and $u'$ do not occur together in any arc of $\boldsymbol{\mathcal{H}}$.*

(iii) *Under any $\mathcal{C}$-excluded colouring, $u$ and $u'$ get different colours.*

*Proof.* Let $\mathcal{G}$ be the $(r+1)$-chromatic $\ell$-graph with girth $\geq g$, which is guaranteed by Theorem 7.3, and let $\mathcal{C}$ be as in the statement of the lemma. Since $\mathcal{C}$ contains all monochromatic $(r,\ell)$-patterns, there is no $\mathcal{C}$-excluded $r$-colouring of the vertices of $\mathcal{G}$ under any orientation of its edges. Let $\boldsymbol{\mathcal{G}}$ be $\mathcal{G}$ with some fixed orientation of the edges. Clearly $\boldsymbol{\mathcal{G}}$ has no $\mathcal{C}$-excluded $r$-colouring. Let $\boldsymbol{\mathcal{H}}_0$ be a subgraph of $\boldsymbol{\mathcal{G}}$ that is minimal with respect to the fact that it has no $\mathcal{C}$-excluded $r$-colouring. Since $\mathcal{C}$ does not contain all $(r,\ell)$-patterns there is a $\mathcal{C}$-excluded $r$-colouring of any graph with a just single arc, so $\boldsymbol{\mathcal{H}}_0$ is non-empty.

Let $(u_1, \ldots, u_\ell)$ be an arc of $\boldsymbol{\mathcal{H}}_0$, and let $v_1, \ldots, v_\ell$ be vertices not in $V(\boldsymbol{\mathcal{H}}_0)$. For $i = 1, \ldots, \ell$, form the oriented $\ell$-graph $\boldsymbol{\mathcal{H}}_i$ from $\boldsymbol{\mathcal{H}}_0$ by removing the arc $(u_1, \ldots, u_\ell)$ and adding the arc $(v_1, \ldots, v_i, u_{i+1}, \ldots, u_\ell)$. Now $\boldsymbol{\mathcal{H}}_\ell$ clearly has a $\mathcal{C}$-excluded colouring, while $\boldsymbol{\mathcal{H}}_0$ does not. Let $i > 0$ be such that $\boldsymbol{\mathcal{H}}_i$ has a $\mathcal{C}$-excluded colouring and $\boldsymbol{\mathcal{H}}_{i-1}$ does not. We show that $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_i$, $u = u_i$ and $u' = v_i$ satisfy the requirements of the lemma. Certainly $\boldsymbol{\mathcal{H}}$ has a $\mathcal{C}$-excluded $r$-colouring, so we have property (i) of the lemma. Since $v_i$ is in only one arc of $\boldsymbol{\mathcal{H}}$, and this arc does not contain $u_i$, we have property (ii) of the lemma. For property (iii), we must now show that under any $\mathcal{C}$-excluded colouring $\chi$ of $\boldsymbol{\mathcal{H}}$, $\chi(u_i) \neq \chi(v_i)$. Assume that $\chi(u_i) = \chi(v_i)$, then $\chi$ is a $\mathcal{C}$-excluded colouring of $\boldsymbol{\mathcal{H}}_{i-1}$. This is a contradiction, so we are done.

□

We are now ready to prove Lemma 2.2.

*Proof.* Let $r \geq 2$ and $H$ be a 3-connected graph.

**a)** We prove a slightly stronger statement than needed. In particular we prove the existence of a negative signal sender $S^- = S^-(r, H, \{x_0, u\}, \{x_0, u'\})$ which has

the added property that the vertices $u$ and $u'$ are not adjacent. The proof will be constructive. Before we can construct $S^-$ we have to define a graph $F$ and an oriented $\ell$-graph $\boldsymbol{\mathcal{H}}$ which we use in the construction.

**The Graph $F$.** Let $F'$ be a graph that is $r$-ramsey-minimal with respect to $H$, and let $F$ be $F'$ with an edge $\{x, y\}$ removed. We claim the following.

$(*)$ Under any $H$-free $r$-colouring of the edges of $F$, $x$ has incident edges of every colour.

Indeed, since $H$ is 3-connected, each of its vertices has degree at least 2. So if $\chi$ were an $H$-free $r$-colouring of $F'$ and $x$ had no incident edge of colour $i$, then we could extend $\chi$ to a $H$-free $r$-colouring of $F'$ by setting $\chi(\{x, y\}) = i$. This would be a contradiction.

**The Oriented $\ell$-graph $\boldsymbol{\mathcal{H}}$.** Let $\ell$ be the degree of the vertex $x$ in $F$. Fix an orientation $(x_1, \dots, x_\ell)$ of the neighbourhood of $x$ in $F$. Let $\mathcal{C}$ be the set of $(r, \ell)$-patterns $(r_1, \dots, r_\ell)$ such that the following holds:

> The $r$-colouring $\chi$ of the edges $\{x, x_1\}, \dots, \{x, x_\ell\}$ of $F$ defined by $\chi(\{x, x_i\}) = r_i$, for all $i$, cannot be extended to an $H$-free $r$-colouring of the edges of $F$.

By statement $(*)$, $\mathcal{C}$ contains all $(r, \ell)$-patterns with less than $r$ colours, in particular, it contains all monochromatic $(r, \ell)$-patterns. Moreover, because $F$ has an $H$-free $r$-colouring, $\mathcal{C}$ is not the set of all $(r, \ell)$-patterns, so we can apply Lemma 7.5. Let $g = |V(H)| + 1$, and let $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\mathcal{C}, r, \ell, g)$ be from Lemma 7.5.

**The Construction of $S^-$.** We now construct the graph $S^- = S^-(r, H, \{x_0, u\}, \{x_0, u'\})$, by taking $|\boldsymbol{\mathcal{H}}|$ copies of $F$ and identifying the neighbourhood of $x$ in each of them with an arc in $\boldsymbol{\mathcal{H}}$, and then identify all copies of $x$. In this way we will be able to relate colourings of the vertices of $\boldsymbol{\mathcal{H}}$, with colourings of the edges of $S^-$. Specifically, construct $S^-$ as follows.

**Construction 7.6.** $(S^- = S^-(r, H, \{x_0, u\}, \{x_0, u'\}))$

(i) For every arc $\boldsymbol{a} = (v_1, \dots, v_\ell)$ in $\boldsymbol{\mathcal{H}}$ let $F^{\boldsymbol{a}}$ be a copy of $F$. The graph $S^-$ is the union of these copies of $F$. All vertices are distinct unless explicitly identified below. Let $x^a$ and $\{x_1^{\boldsymbol{a}}, \dots, x_\ell^{\boldsymbol{a}}\}$ denote the copies of $x$ and its neighbourhood $\{x_1, \dots, x_\ell\}$ in $F^{\boldsymbol{a}}$.

(ii) For every arc $\boldsymbol{a} = (v_1, \dots, v_\ell)$ in $\boldsymbol{\mathcal{H}}$ and every $i = 1, \dots, \ell$, identify the vertex $x_i^{\boldsymbol{a}}$ of $F^{\boldsymbol{a}}$ with $v_i$ of $\boldsymbol{\mathcal{H}}$.

(iii) Identify the vertices $x^{\boldsymbol{a}}$ for all $\boldsymbol{a} \in \boldsymbol{\mathcal{H}}$ and call the resulting vertex $x_0$. As a result of this identification, we have that $N(x_0) = V(\boldsymbol{\mathcal{H}})$ in $S^-$.

By property (ii) of Lemma 7.5, the vertices $u$ and $u'$ do not occur together in any arc of $\boldsymbol{\mathcal{H}}$. Thus they are not adjacent in the graph $S^-$ just constructed. For statement a) of the lemma, we have left to prove that $S^- = S^-(r, H, \{x_0, u\}, \{x_0, u'\})$ is a negative signal sender. The proof consists of the following three claims, the first being used to prove the second.

**Claim 7.7.** *Any copy of $H$ in $S^-$ is entirely within $F^{\boldsymbol{a}}$ for some $\boldsymbol{a} \in \boldsymbol{\mathcal{H}}$.*

*Proof.* Let $H_0$ be a copy of $H$ in $S^-$. For $\boldsymbol{\mathcal{A}} \subset \boldsymbol{\mathcal{H}}$ let $F^{\boldsymbol{\mathcal{A}}}$ be the subgraph of $S^-$ induced by the vertex set $\bigcup_{\boldsymbol{a} \in \boldsymbol{\mathcal{A}}} V(F^{\boldsymbol{a}}) \subset V(S^-)$. Let $\boldsymbol{\mathcal{A}}^*$ be a minimum subgraph

of $\boldsymbol{\mathcal{H}}$ for which $H_0 \subset F^{\boldsymbol{\mathcal{A}^*}}$. We will show that $|\boldsymbol{\mathcal{A}^*}| = 1$. Towards contradiction, assume that $|\boldsymbol{\mathcal{A}^*}| > 1$. By the minimality of $\boldsymbol{\mathcal{A}^*}$, $|\boldsymbol{\mathcal{A}^*}| \leq |V(H_0)| = g - 1$, so being a subgraph of $\boldsymbol{\mathcal{H}}$, $\boldsymbol{\mathcal{A}^*}$ contains no circuits. This implies that there exists $\boldsymbol{a_0} \in \boldsymbol{\mathcal{A}^*}$ such that $\boldsymbol{a_0}$ intersects the rest of $\boldsymbol{\mathcal{A}^*}$ in at most one vertex, i.e., such that $|V(\boldsymbol{a_0}) \cap V(\boldsymbol{\mathcal{A}^*} - \boldsymbol{a_0})| \leq 1$.

If $V(\boldsymbol{a_0}) \cap V(\boldsymbol{\mathcal{A}^*} - \boldsymbol{a_0})$ is empty, then removing $x_0$ (see Construction 7.6 (iii),) disconnects $F^{\boldsymbol{a_0}}$ from the rest of $F^{\boldsymbol{\mathcal{A}^*}}$. But by the minimality of $F^{\boldsymbol{\mathcal{A}^*}}$, the intersection of $V(H_0)$ and $V(F^{\boldsymbol{a_0}}) - \{x_0\}$ is non-trivial, so this disconnects $H_0$.

On the other hand, if $V(\boldsymbol{a_0}) \cap V(\boldsymbol{\mathcal{A}^*} - \boldsymbol{a_0})$ contains some vertex $y$, then $y$ is adjacent to $x_0$ and the edge $\{x_0, y\}$ is in $F^{\boldsymbol{a_1}}$ for some other $\boldsymbol{a_1} \in \boldsymbol{\mathcal{A}^*}$. Removing $x_0$ and $y$ disconnects $F^{\boldsymbol{a_0}}$ from the rest of $F^{\boldsymbol{\mathcal{A}^*}}$, but by the minimality of $\boldsymbol{\mathcal{A}^*}$, the intersection of $V(H_0)$ and $V(F^{\boldsymbol{a_0}}) - \{x_0, y\}$ is non-trivial, so this disconnects $H_0$.

Either way we have disconnected $H_0$ by removing at most two vertices. This contradicts the fact that $H_0$ is 3-connected. Thus $|\boldsymbol{\mathcal{A}^*}| = 1$ and the claim is proven. □

**Claim 7.8.** *There exists an $H$-free $r$-colouring of $S^-$.*

*Proof.* Let $\chi'$ be a $\mathcal{C}$-excluded $r$-colouring of the vertices of $\boldsymbol{\mathcal{H}}$. For each vertex $v$ in $\boldsymbol{\mathcal{H}}$, let $\chi(\{x_0, v\}) = \chi'(v)$. For each copy $F^{\boldsymbol{a}}$ of $F$ this gives a colouring of the edges $(\{x_0, x_1^{\boldsymbol{a}}\}, \ldots, \{x_0, x_\ell^{\boldsymbol{a}}\})$ which is a pattern not in $\mathcal{C}$. Thus we can extend the $r$-colouring $\chi$ to the edges of $F^{\boldsymbol{a}}$ in such a way that there are no monochromatic copies of $H$ in $F^{\boldsymbol{a}}$. By Claim 7.7, there are no other copies of $H$ in $S^-$, and so $\chi$ is an $H$-free $r$-colouring of $S^-$. □

**Claim 7.9.** *Under any $H$-free $r$-colouring of $S^-$, the edges $\{x_0, u\}$ and $\{x_0, u'\}$ get different colours.*

*Proof.* Assume that we are given $H$-free $r$-colouring $\chi$ of $S^-$. This induces an $r$-colouring $\chi'$ of $V(\boldsymbol{\mathcal{H}})$ by $\chi'(v) = \chi(\{x_0, v\})$. For any arc $\boldsymbol{a} = (v_1, \ldots, v_\ell)$ in $\boldsymbol{\mathcal{H}}$ this has the pattern

$$(\chi'(v_1), \ldots, \chi'(v_\ell)) = (\chi(\{x_0, v_1\}), \ldots, \chi(\{x_0, v_\ell\})) = (\chi(\{x_0, x_1^{\boldsymbol{a}}\}), \ldots, \chi(\{x_0, x_\ell^{\boldsymbol{a}}\})),$$

which because $\chi$ is an $H$-free $r$-colouring of $F^{\boldsymbol{a}}$, is a pattern not in $\mathcal{C}$. Thus $\chi'$ is a $\mathcal{C}$-excluded colouring of $\boldsymbol{\mathcal{H}}$. But any $\mathcal{C}$-excluded colouring $\chi'$ of $\boldsymbol{\mathcal{H}}$ has the property that $\chi'(u) \neq \chi'(u')$. Thus $\chi(\{x_0, u\}) \neq \chi(\{x_0, u'\})$. □

We have shown that $S^-$ is a negative signal sender $S^-(r, H, \{x_0, u\}, \{x_0, u'\})$ in which the vertices $u$ and $u'$ are not adjacent. This is a stronger statement than a) of the lemma. This stronger statement will be used in the proof of part c)

**b)** We prove a similar strengthening of statement b) of the lemma. Specifically, we prove the existence of a positive signal sender, $S^+(r, H, \{v_0, v_1\}, \{v_0, v_t\})$ in which the vertices $v_1$ and $v_t$ are not adjacent.

We first prove the case $H = K_k$ for any $k \geq 3$, and then sketch the extension of the proof to the case of a general 3-connected graph $H$. In both cases, the proof consists of constructing the graph $S^+ = S^+(r, H, \{v_0, v_1\}, \{v_0, v_t\})$, observing that it has the property that the vertices $v_1$ and $v_t$ are not adjacent, and then proving that it is a positive signal sender by proving analogues of Claims 7.7, 7.8, and 7.9.

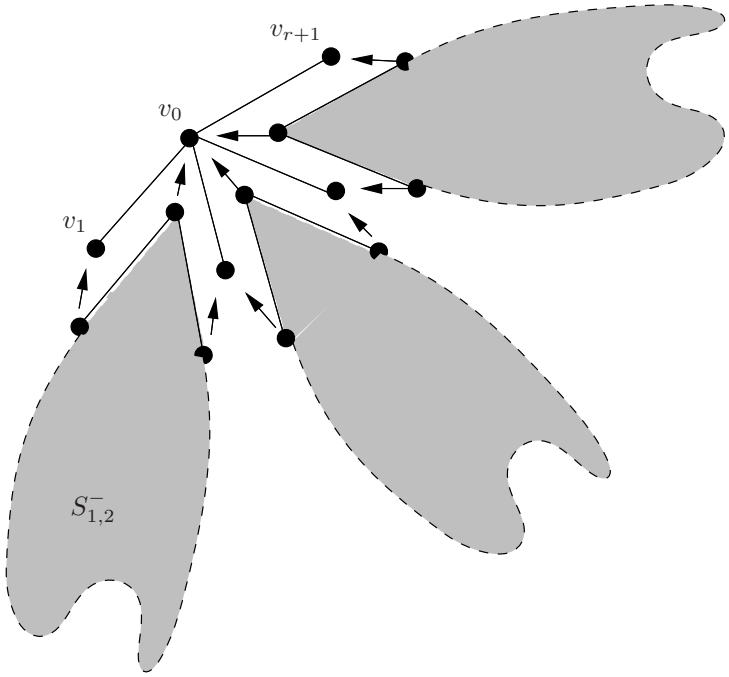


FIGURE 7. A subgraph of $S^+$

In the case that $H = K_k$, we will have $t = r+1$, so our construction will provide the signal sender $S^+ = S^+(r, H, \{v_0, v_1\}, \{v_0, v_{r+1}\})$ in which the vertices $v_1$ and $v_{r+1}$ are not adjacent.

**The Construction of $S^+$ in the case $H = K_k$.** Take the star with root $v_0$ and leaves $v_1, \dots, v_{r+1}$. For each pair $1 \le i < j \le r+1$ except $i = 1$ and $j = r+1$, let $S^-_{i,j} = S^-(r, K_k, \{v_0, v_i\}, \{v_0, v_j\})$ be a copy of $S^-(r, K_k, \{x_0, u\}, \{x_0, u'\})$ in which we identify the the copies of $x_0, u$, and $u'$ with the vertices $v_0, v_i$, and $v_j$, respectively. Let the $\binom{r+1}{2} - 1$ copies of $S^-(r, K_k, \{x_0, u\}, \{x_0, u'\})$ be otherwise vertex disjoint, and let $S^+$ be their union. (See Figure 7).

By construction, it is clear that the vertices $v_1$ and $v_{r+1}$ are not adjacent. To finish the proof of part b) for the case that $H = K_k$ is a complete graph, we show that $S^+$ is a positive signal sender. We do this by proving the following claim, the parts of which are analogues of the Claims 7.7, 7.8, and 7.9.

**Claim 7.10.** *The graph $S^+$ has the following properties.*

(i) *Any copy of $K_k$ in $S^+$ occurs within $S^-_{i,j}$ for some $1 \le i < j \le r+1$.*
(ii) *There exists a $K_k$-free $r$-colouring of $S^+$.*
(iii) *Under any $K_k$-free $r$-colouring of $S^+$, the edges $\{v_0, v_1\}$ and $\{v_0, v_{r+1}\}$ get the same colour.*

*Proof.* **(i)** Since each $S^-_{i,j} = S^-(r, K_k, \{v_0, v_i\}, \{v_0, v_j\})$ had the property that the vertices $v_i$ and $v_j$ were non-adjacent, the vertices $v_1, \dots, v_{r+1}$ form an independent set. Thus any copy $K$ of $K_k$ in $S^+$ must contain at least one vertex $x \in V(S^+) - \{v_0, v_1, \dots v_{r+1}\}$. This vertex $x$ is in $V(S^-_{i,j}) - \{v_0, v_i, v_j\}$ for some

$1 \leq i < j \leq r+1$. But $v_0, v_i, v_j$ are the only vertices of $S^-_{i,j}$ that have neighbours in $S^+$ that are outside of $S^-_{i,j}$. Thus $K \subset S^-_{i,j}$.

**(ii)** Define a $K_k$-free $r$-colouring $\chi$ of $S^+$ as follows. Let $\chi(\{v_0, v_{r+1}\}) = 1$ and $\chi(\{v_0, v_i\}) = i$ otherwise. For each of the $S^-_{i,j}$, $\chi$ gives different colours to the distinguished edges $\{v_0, v_i\}$ and $\{v_0, v_j\}$, so can be extended to a $K_k$-free $r$-colouring of $S^-_{i,j}$. By property (i), this yields a $K_k$-free colouring of $S^+$.

**(iii)** Under any $K_k$-free $r$-colouring of $S^+$, every pair of edges $\{v_0, v_i\}$ and $\{v_0, v_j\}$, for $1 \leq i < j \leq r+1$, except $\{v_0, v_1\}$ and $\{v_0, v_{r+1}\}$, gets distinct colours. Thus $\{v_0, v_1\}$ and $\{v_0, v_{r+1}\}$ get the same colour. □

This concludes the proof of the claim, and so the proof of part b) of the lemma in the particular case that $H = K_k$. We now extend this proof to the general case by constructing a positive signal sender $S^+ = S^+(r, H, \{v_0, v_1\}, \{v_0, v_t\})$, for any 3-connected graph $H$, in which the vertices $v_1$ and $v_t$ are not adjacent. In extending the proof from the case where $H = K_k$, we have to be more careful to ensure the analogue of Claim 7.10(i).

**Construction of $S^+$ in the case that $H$ is 3-connected.** Begin with an auxillary $(r+1)$-critical graph $G'$ of girth $g > |V(H)|$ with vertices $V(G') = \{v_1, \ldots, v_t\}$. Without loss of generality, assume that $\{v_1, v_t\}$ is an edge of $G'$ and let $G = G' - \{v_1, v_t\}$. Notice that under any proper $r$-colouring $\psi$ of the vertices of $G$, we have that $\psi(v_1) = \psi(v_t)$.

Construct $S^+$ as follows. Let $v_0$ be a vertex. For every edge $\{v_i, v_j\} \in G$, let $S^-_{i,j} = S^-(r, H, \{v_0, v_i\}, \{v_0, v_j\})$ be a copy of $S^-(r, H, \{x_0, u\}, \{x_0, u'\})$. Other than on the vertices $v_0, v_1, \ldots, v_t$ let these copies of $S^-(r, K_k, \{x_0, u\}, \{x_0, u'\})$ be vertex disjoint. That is, let $V(S^-_{i,j}) \cap V(S^-_{i',j'})$ be $\{v_0\}$ if $i \neq i'$ and $j \neq j'$, $\{v_0, v_i\}$ if $i = i'$ and $j \neq j'$, and $\{v_0, v_j\}$ if $i \neq i'$ and $j = j'$. Let $S^+$ be the union of these $S^-_{i,j}$.

By the construction, the the vertices $v_1$ and $v_t$ are non-adjacent in $S^+$. We thus have just to prove that $S^+ = S^+(r, h, \{v_0, v_1\}, \{v_0, v_t\})$ is a positive signal sender. We do with the following claim.

**Claim 7.11.** *The graph $S^+$ has the following properties.*

(i) *Any copy of $H$ in $S^+$ occurs within $S^-_{i,j}$ for some $\{v_i, v_j\} \in G$.*
(ii) *There exists an $H$-free $r$-colouring of $S^+$.*
(iii) *Under any $H$-free $r$-colouring of $S^+$, the edges $\{v_0, v_1\}$ and $\{v_0, v_t\}$ get the same colours.*

*Sketch of Proof.* Property (i) follows from the fact that we started with a girth $g > |V(H)|$ graph $G'$, and from the 3-connectedness of $H$. The proof is a simpler version of the proof of Claim 7.7.

The proofs of properties (ii) and (iii) are analogous to the corresponding properties in the $K_k$ case.

Indeed, for property (ii), take an $r$-colouring of the vertices of $G$. This induces an $r$-colouring $\chi$ of the edges of $S^+$ such that the edges $\{v_0, v_i\}$ and $\{v_0, v_j\}$ of $S^-_{i,j}$ get distinct colours for each $\{v_i, v_j\} \in G$. We can therefore extend $\chi$ to an $H$-free

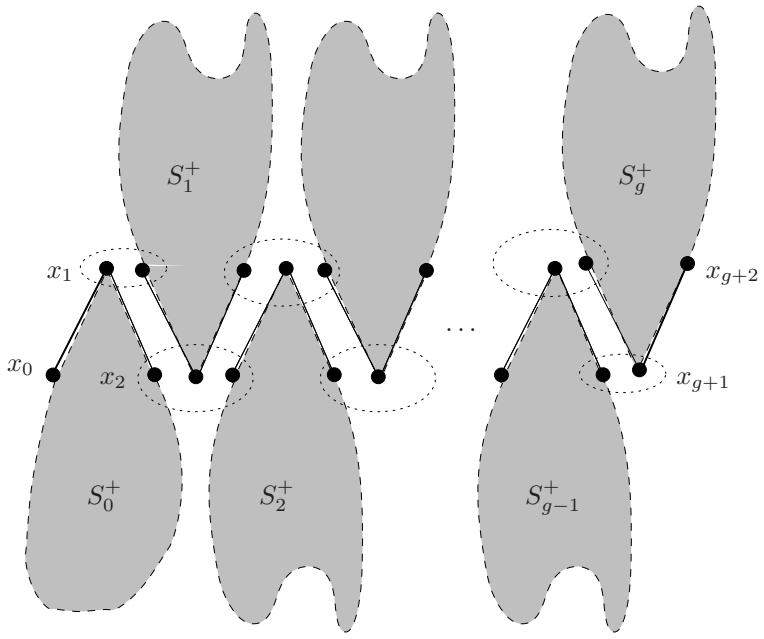


FIGURE 8. $S^+$ where distinguished edges are distance $g$ apart.

$r$-colouring of each of the $S^-_{i,j}$. By property (i), $\chi$ is then an $H$-free $r$-colouring of $S^+$.

For property (iii), take an $H$-free $r$-colouring $\chi$ of $S^+$. The edges $\{v_0, v_i\}$ and $\{v_0, v_j\}$ get different colours for any edge $\{v_i, v_j\} \in G$. Thus $\chi$ induces a proper $r$-colouring $\psi$ of $V(G)$ by $\psi(v_i) = \chi(\{v_0, v_i\})$. But then $\psi(v_1) = \psi(v_t)$, so $\chi(\{v_0, v_1\}) = \chi(\{v_0, v_t\})$.

□

This completes the proof of the claim, and so the proof of part b) of the lemma.

**c)** Fix an integer $g > 0$. We now prove that there exists a positive signal sender $S^+ = S^+(r, H, \{x_0, x_1\}, \{x_{g+1}, x_{g+2}\})$, and that there exists a negative signal sender $S^- = S^-(r, H, \{x_0, x_1\}, \{x_{g+1}, x_{g+2}\})$, such that in each of them, the edges $\{x_0, x_1\}$ and $\{x_{g+1}, x_{g+2}\}$ are distance $g$ apart. Again, the proofs are constructive.

**The Construction of $S^+$.** For $i = 0 \dots g$, let $S_i^+ = S^+(r, H, \{x_i, x_{i+1}\}, \{x_{i+1}, x_{i+2}\})$ be a copy of $S^+(r, H, \{v_0, v_1\}, \{v_0, v_t\})$ from part b) of the lemma. Let $S^+$ be the union of these $g+1$ graphs where all vertices except $x_1, \dots, x_{g+1}$ are distinct. (See Figure 8).

Observe that $S^+$ contains the path $x_1, \dots, x_{g+1}$ of length $g$ from $\{x_0, x_1\}$ to $\{x_{g+1}, x_{g+2}\}$. Since the vertices $x_i$ and $x_{i+2}$ were not adjacent in $S_i^+$, for $i = 0, \dots, g$, and the only edges of $S^+$ are from the graphs $S_i^+$, this is the shortest path between $\{x_0, x_1\}$ and $\{x_{g+1}, x_{g+2}\}$. Thus these edges are distance $g$ apart.

To show that $S^+ = S^+(r, H, \{x_0, x_1\}, \{x_{g+1}, x_{g+2}\})$ is a positive signal sender, we prove the following claim.

**Claim 7.12.** *The graph $S^+$ has the following properties.*

(i) *Any copy of $H$ in $S^+$ occurs within $S_i^+$ for some $0 \leq i \leq g$.*
(ii) *There exists a $H$-free $r$-colouring of $S^+$.*
(iii) *Under any $H$-free $r$-colouring of $S^+$, the edges $\{x_0, x_1\}$ and $\{x_{g+1}, x_{g+2}\}$ get the same colour.*

*Sketch of Proof.* The proof of property (i) is very similar to the proof of property (i) of Claim 7.10.

For property (ii), observe that any colouring that is constant on the edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{g+1}, v_{g+2}\}$ can be extended to a $H$-free $r$-colouring of $S^+$.

For property (iii), observe that any $H$-free $r$-colouring of $S^+$ is constant on the edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{g+1}, v_{g+2}\}$. □

**The Construction of $S^-$.** Use exactly the same construction as for $S^+$ above, except replace the first positive sender, $S_0^+ = S^+(r, H, \{x_0, x_1\}, \{x_1, x_2\})$ with the negative sender $S^-(r, H, \{x_0, x_1\}, \{x_1, x_2\})$. Let $S^-$ be the resulting graph.

The proof that $S^-$ is a negative signal sender in which the the edges $\{x_0, x_1\}$ and $\{x_{g+1}, x_{g+2}\}$ are distance $g$ apart is very similar to the proof above for $S^+$.

This completes the proof of the lemma.

□

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